Weighted counting of inversions on alternating sign matrices

Masato Kobayashi

Kanagawa University

September 19, 2019

Keywords

- **bigrassmannian permutations**
- **Bruhat order**
- **inversion**
- **alternating sign matrix (ASM)**

This talk is all about

two formulas for enumeration of bigrassmannian permutations over permutations and ASMs by inversions.

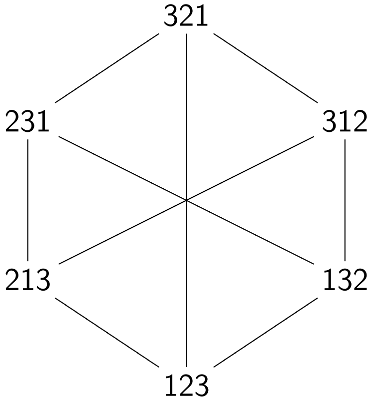
Def. (i, j) is an **inversion** of a permutation w of $\{1, 2, \ldots, n\}$ if $i < j$ and $w(i) > w(j)$. The **inversion statistic** of w is the number of such pairs

$$\ell(w) = |\{(i, j) \mid i < j \text{ and } w(i) > w(j)\}|.$$

Example:

$$w = 321$$

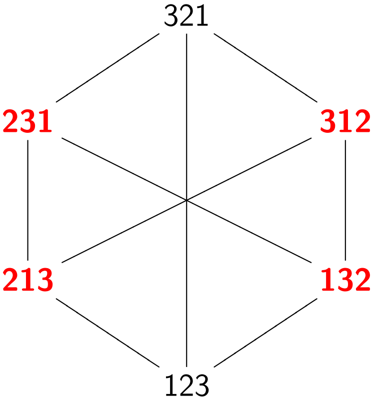
Inversion: $(1,2),(1,3),(2,3)$
$\Longrightarrow \ell(w) = 3.$



inversion 3

Def

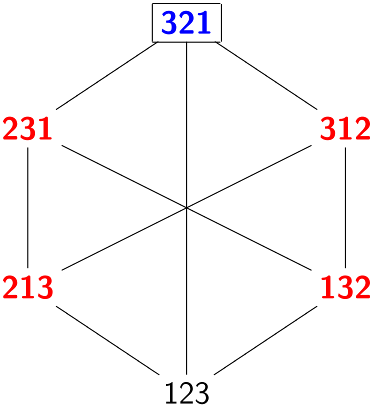
Say a permutation $w \in S_n$ is **bigrassmannian** if there exists a unique pair (i, j) such that $w^{-1}(i) > w^{-1}(i+1)$ and $w(j) > w(j+1)$.


321
231
312
213
132
123

The **bigrassmannian** statistic for w is

$$\beta(w) = |\{v \in S_n \mid v \leq w \text{ and } v \text{ is bigrassmannian}\}|$$

with $\leq$ Bruhat order.



$$\beta(321) = 4.$$

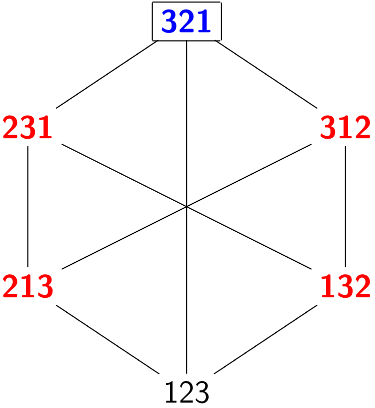
Thm 1 (Kobayashi 2011)

$$\beta(w) = \sum_{i<j, w(i)>w(j)} (w(i) - w(j))$$

for each $w \in S_n$.

We can interpret this formula as weighted counting of inversions.

$$\beta(321) = (3-2) + (3-1) + (2-1) = 4.$$



Extend this formula:
permutations $\rightarrow$ Alternating sign matrices

$A = (a_{ij})$: $n \times n$ matrix
Def: A is an **alternating sign matrix** (ASM) if for all i, j, we have

$$a_{ij} \in \{-1, 0, 1\}, \qquad \sum_{k=1}^{j} a_{ik} \in \{0, 1\},$$

$$\sum_{k=1}^{i} a_{kj} \in \{0, 1\} \quad \text{and} \quad \sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{kj} = 1.$$

$\mathcal{A}_n :=$ the set of all alternating sign matrices of size n.

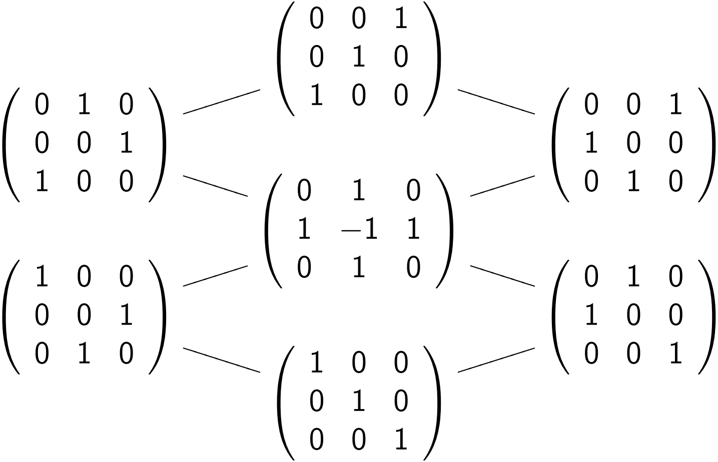
Examples of ASM.

$$321 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Fact: there exists an exntension of Bruhat order onto $\mathcal{A}_n$ such that
$(\mathcal{A}_n, \leq)$ is MacNeille completion of $(S_n, \leq)$.
$\Longrightarrow \mathcal{A}_n$ is a distributive lattice (graded).

$\mathcal{A}_3$


$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Def **bigrassmannian statistics** for ASMs:

$$\beta(B) = |\{A \in \mathcal{A}_n \mid A \leq B \text{ and } A \text{ is bigrassmannian}\}|.$$

cf.

$$\beta(w) = |\{v \in S_n \mid v \leq w \text{ and } v \text{ is bigrassmannian}\}|$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Def We say that (i, j, k, l) is an **inversion** of A if $i < j, k < l$ and $a_{jk}a_{il} \neq 0$. Also, let us say that $l - k$ is the **weight** of this inversion. Define the **inversion statistic** of A by

$$I(A) = \sum_{i<j,k<l} a_{jk}a_{il}.$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & \mathbf{1} \\ 0 & \mathbf{1} & 0 \end{pmatrix}$$

an inversion of weight 1.

$$\begin{pmatrix} 0 & 0 & \mathbf{1} \\ 0 & 1 & 0 \\ \mathbf{1} & 0 & 0 \end{pmatrix}$$

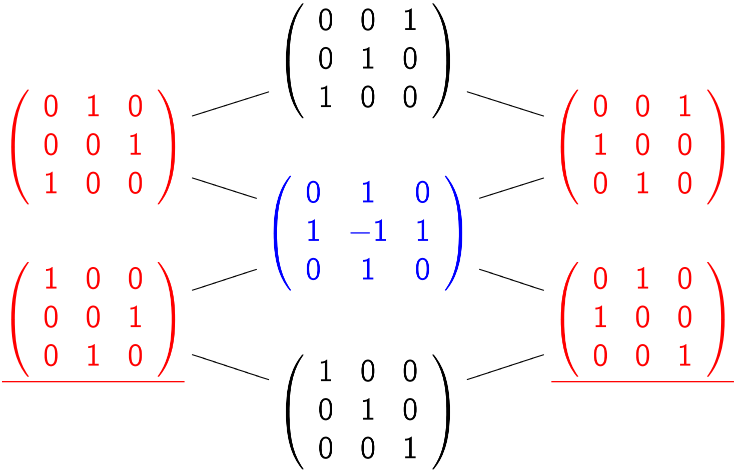
an inversion of weight 2.

Thm 2 (Kobayashi 2019)
For each $A \in \mathcal{A}_n$, we have:

$$\beta(A) = \sum_{i<j, k<l} (l-k) a_{jk} a_{il}.$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\beta(A) = 1 \cdot 1 + 1 \cdot 1 = 2.$$



Further research

- Find $\displaystyle\sum_{A \in \mathcal{A}_n} q^{I(A)}$.
- What about type B, D, Affine?

References

- Kobayashi, Enumeration of bigrassmannian permutations below a permutation in Bruhat order, Order 28 (2011), 131-137.
- Kobayashi, Weighted counting of inversions on alternating sign matrices, math arXiv:1904.02265.

Thanks!