MÖBIUS FUNCTIONS OF HIGHER RANK AND DIRICHLET SERIES

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Abstract. We introduce Möbius functions of higher rank, a new class of arithmetic functions, so that the classical Möbius function is of rank 2. With this idea, we evaluate Dirichlet series on the sum of reciprocal square of all r-free numbers. For the proof, Riemann zeta function and cyclotomic polynomials play a key role.

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Date: March 14, 2017.
2010 *Mathematics Subject Classification.* Primary:11A25; Secondary:11M32.
Key words and phrases. arithmetic function, cyclotomic polynomial, Dirichlet series, Möbius function, Riemann zeta function.
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1. Introduction

1.1. **Classical Möbius and zeta functions.** The *Möbius function* plays an important role in number theory. Its definition is simple: $\mu(n) = 1$ and

$$\mu(n) = \begin{cases} (-1)^k & n = p_1 \cdots p_k, \text{ primes } p_j \text{ all distinct,} \\ 0 & p^2 \mid n \text{ for some prime } p. \end{cases}$$

Riemann zeta function is also another important topic in number theory. It is an analytic function of complex variable s (pole at 1):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1.$$

It has an infinite product (known as the *Euler product*) expression:

$$\zeta(s) = \prod_{p:\text{prime}} (1 - p^{-s})^{-1}, \quad \mathrm{Re}(s) > 1.$$

See Titchmarsh [3] for more details. Table 1 shows the zeta values at positive even integers up to 20.

Table 1. zeta values at even positive integers

$2n$	$\zeta(2n)$	$2n$	$\zeta(2n)$
2	$\frac{\pi^2}{6}$	12	$\frac{691\pi^{12}}{638512875}$
4	$\frac{\pi^4}{90}$	14	$\frac{2\pi^{14}}{18243225}$
6	$\frac{\pi^6}{945}$	16	$\frac{3617\pi^{16}}{325641566250}$
8	$\frac{\pi^8}{9450}$	18	$\frac{43867\pi^{18}}{38979295480125}$
10	$\frac{\pi^{10}}{93555}$	20	$\frac{174611\pi^{20}}{1531329465290625}$

There is a deep relation between the Möbius and zeta functions; we can "invert" $\zeta(s)$:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \mathrm{Re}(s) > 1.$$

For instance, when $s = 2$, we obtain the inverse of Euler's work $\zeta(2) = \pi^2/6$ as

$$\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right)^{-1} = \frac{1}{\zeta(2)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \cdots.$$

1.2. **Main results.** In this article, we introduce Möbius functions of higher rank, a new class of arithmetic functions, so that the classical Möbius function is of rank 2.

$$\begin{tikzcd} \boxed{\text{classical Möbius function } \mu = \mu_2} \arrow[d] \\ \boxed{\text{Möbius functions of higher rank } \mu_r \ (r = 1, 2, \ldots, \infty)} \end{tikzcd}$$

For a positive integer r, say n is *r-free* if there exists some prime p such that p^r divides n; thus, 2-free is square-free and 3-free is cube-free as usually said. We will see that μ_r is similar to $\mu = \mu_2$: $\mu_r(n) \neq 0$ if and only if n is r-free (Section 3).

The main result of this article is to evaluate several Dirichlet series

$$\sum_{n:r\text{-free}} \frac{\mu_r(n)}{n^s}$$

with $r \in \{3, 4, 5\}$ and $s \in \{2, 3\}$.

Theorem 1.1 ($s = 2$)**.** *The following equalities hold:*

(1) $\displaystyle\sum_{n:3\text{-free}} \frac{\mu_3(n)}{n^2} = \frac{45045}{691\pi^4}.$

(2) $\displaystyle\sum_{n:4\text{-free}} \frac{\mu_4(n)}{n^2} = \frac{630}{\pi^6}.$

(3) $\displaystyle\sum_{n:5\text{-free}} \frac{\mu_5(n)}{n^2} = \frac{1091215125}{174611\pi^8}.$

Theorem 1.2 ($s = 3$)**.**

$$\left(\sum_{n:3\text{-free}} \frac{1}{n^3}\right)\left(\sum_{n:3\text{-free}} \frac{\mu_3(n)}{n^3}\right) = \frac{41247931725}{43867\pi^{12}}.$$

Here, μ_3, μ_4, μ_5 are the Möbius functions of rank 3, 4, 5 respectively. For the proofs, it is key to understand interactions of the following three concepts:

- Euler product for Riemann zeta function
- Möbius functions of higher rank
- Cyclotomic polynomials

We will give these details later.

Additional results: it is possible to generalize the "Möbius inversion formula" to higher rank: for $n = p_1^{m_1} \cdots p_k^{m_k}$, the prime factorization of n, let

$$m_*(n) = |\{j \mid m_j \equiv 3, 4 \bmod 6\}|.$$

Theorem 1.3 (Möbius inversion of rank 3)**.**

$$\left(\sum_{n=1}^{\infty} \frac{\mu_3(n)}{n^s} \right)^{-1} = \sum_{\substack{n=p_1^{m_1} \cdots p_k^{m_k} \\ m_j \not\equiv 2,5 \bmod 6}} \frac{(-1)^{m_*(n)}}{n^s}.$$

As a by-product, we get a new expression of π:

$$\pi = \left(\frac{45045}{691} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{10^2} + \cdots \right) \right)^{1/4}.$$

1.3. **Notation.**

- Let $\mathbf{N}$ denote the set of positive integers. In addition, $\mathbf{N}^2$ means the set of square numbers $\{1^2, 2^2, 3^2, \ldots\}$.
- Often, writing
$$n = p_1^{m_1} \cdots p_k^{m_k}$$
means the factorization of n into *distinct* prime numbers ($p_i \neq p_j$ for $i \neq j$) with each m_j positive unless otherwise specified.
- $d \mid n$ means d divides n.
- $\prod_p$ indicates an infinite product over all primes p.

2. Preliminaries

Let us begin with recalling some fundamental definitions and facts on arithmetic functions; you can find this topic in a standard textbook on number theory as Apostol [2]. We thus omit most of the proofs here.

2.1. **Arithmetic functions.** An *arithmetic function* is a map

$$f : \mathbf{N} \to \mathbb{C}.$$

Example 2.1.

- **Möbius function:**
$$\mu(n) = \begin{cases} 1 & n = 1, \\ (-1)^k & n = p_1 \cdots p_k, \text{ primes } p_j \text{ all distinct}, \\ 0 & p^2 \mid n \text{ for some prime } p. \end{cases}$$
- **Omega function:** For $n = p_1^{m_1} \cdots p_k^{m_k}$ with p_j primes,
$$\Omega(n) = m_1 + \cdots + m_k.$$

- **Liouville function:** $\lambda(n) = (-1)^{\Omega(n)}$.
- **Characteristic function:** For a subset $A \subseteq \mathbf{N}$,
$$\chi_A(n) = \begin{cases} 1 & n \in A, \\ 0 & n \notin A. \end{cases}$$
In particular, $|\mu(n)|$ is a characteristic function of the set of 2-free numbers.
- **Constant function:** $1(n) = 1$ for all n.
- **unit function:** $u(n) = \begin{cases} 1 & n = 1, \\ 0 & n \neq 1. \end{cases}$

We say that an arithmetic function f is *multiplicative* if $f(1) = 1$ and
$$f(mn) = f(m)f(n) \quad \text{whenever} \quad \gcd(m,n) = 1.$$
It is easy to check that $\mu, \lambda, 1, u$ are all multiplicative.

Table 2. Arithmetic functions

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\mu(n)$	1	−1	−1	0	−1	1	−1	0	0	1	$\cdots$
$\Omega(n)$	0	1	1	2	1	2	1	3	2	2	$\cdots$
$\lambda(n)$	1	−1	−1	1	−1	1	−1	−1	1	1	$\cdots$
$1(n)$	1	1	1	1	1	1	1	1	1	1	$\cdots$
$u(n)$	1	0	0	0	0	0	0	0	0	0	$\cdots$

2.2. **Dirichlet series.** For two arithmetic functions f and g, define the *Dirichlet product* $f * g$ by
$$(f * g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right).$$
The unit function u satisfies
$$f * u = u * f = f$$
for all arithmetic functions f. If $f * g = g * f = u$, then we write $g = f^{-1}$ and call it the *Dirichlet inverse* of f; assuming $f(1) \neq 0$, there exists f^{-1}.

Fact 2.2. Let f and g be arithmetic functions. Suppose they are multiplicative. Then, so are $f * g$ and f^{-1}.

In this way, multiplicative functions form a group and u is indeed a group-theoretic unit.

Remark 2.3. If multiplicative functions f, g satisfy

$$f(p^m) = g(p^m) \quad \text{for all primes } p \text{ and } m \geq 1,$$

then $f(n) = g(n)$ for all $n \in \mathbf{N}$. Hence, to determine a multiplicative function, it is enough to know values only at prime powers.

A *Dirichlet series* for f is a series in the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

for a complex number s (in this article, we deal with only $s = 2, 3$ and convergent series). Riemann zeta function is an example of such series with $f(n) = 1(n) = 1$.

Observe that

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s}$$

for all f, g. Then classical results

$$(\mu * 1)(n) = \sum_{d \mid n} \mu(d) = \begin{cases} 1 & n = 1, \\ 0 & n \neq 1 \end{cases}$$

and

$$(\lambda * 1)(n) = \chi_{\mathbf{N}^2}(n) = \begin{cases} 1 & n = N^2 \text{ for some } N, \\ 0 & \text{otherwise} \end{cases}$$

imply the following:

Fact 2.4.

(1) $$\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = 1.$$

(2) $$\left(\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \sum_{n:\,\text{square}} \frac{1}{n^s} = \sum_{N=1}^{\infty} \frac{1}{(N^2)^s} = \zeta(2s).$$

As a consequence, when $s = 2$, we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^2} = \frac{\zeta(4)}{\zeta(2)} = \frac{\pi^2}{15}.$$

Once we introduce the Möbius functions of higher rank μ_r in the next section, we can regard these as extremal cases at $r = 2$ and $r = \infty$ (as shown in Table 3):

$$\sum_{n:2\text{-free}} \frac{\mu_2(n)}{n^2} = \frac{6}{\pi^2} \quad \text{and} \quad \sum_{n:\infty\text{-free}} \frac{\mu_\infty(n)}{n^2} = \frac{\pi^2}{15}.$$

TABLE 3. Main results ($s = 2$)

series	value	zeta expression	factor of Euler product
$\sum_{n:2\text{-free}} \frac{\mu_2(n)}{n^2}$	$\frac{6}{\pi^2}$	$\frac{1}{\zeta(2)}$	$1 - p^{-2}$
$\sum_{n:3\text{-free}} \frac{\mu_3(n)}{n^2}$	$\frac{45045}{691\pi^4}$	$\frac{\zeta(4)\zeta(6)}{\zeta(2)\zeta(12)}$	$1 - p^{-2} + p^{-4}$
$\sum_{n:4\text{-free}} \frac{\mu_4(n)}{n^2}$	$\frac{630}{\pi^6}$	$\frac{\zeta(4)}{\zeta(2)\zeta(8)}$	$1 - p^{-2} + p^{-4} - p^{-6}$
$\sum_{n:5\text{-free}} \frac{\mu_5(n)}{n^2}$	$\frac{1091215125}{174611\pi^8}$	$\frac{\zeta(4)\zeta(10)}{\zeta(2)\zeta(20)}$	$1 - p^{-2} + p^{-4} - p^{-6} + p^{-8}$
$\cdots$		$\cdots$	
$\sum_{n:\infty\text{-free}} \frac{\mu_\infty(n)}{n^2}$	$\frac{\pi^2}{15}$	$\frac{\zeta(4)}{\zeta(2)}$	$1 - p^{-2} + p^{-4} - p^{-6} + p^{-8} - \cdots$

3. Möbius functions of higher rank

For each natural number r or "$r = \infty$", define an arithmetic function

$$\mu_r : \mathbf{N} \to \{-1, 0, 1\}$$

by

$$\mu_r(n) = \begin{cases} 1 & n = 1, \\ (-1)^{m_1 + \cdots + m_k} & n = p_1^{m_1} \cdots p_k^{m_k}, \text{all } m_j < r, \\ 0 & p^r \mid n \text{ for some prime } p. \end{cases}$$

For $r = \infty$, we understand that $m_j < \infty$ always holds and $p^\infty \mid n$ never happens.

Example 3.1. $r = 1$: This is just the unit function.

$$\mu_1(n) = u(n) = \begin{cases} 1 & n = 1, \\ 0 & n \neq 1 \text{ (that is, } p^1 | n \text{ for some prime } p). \end{cases}$$

$r = 2$: the classical Möbius function.

$$\mu_2(n) = \mu(n) = \begin{cases} (-1)^k & n = p_1 p_2 \cdots p_k, \\ 0 & p^2 \mid n \text{ for some prime } p. \end{cases}$$

$r = 3$:

$$\mu_3(n) = \begin{cases} (-1)^{m_1+\cdots+m_k} & n = p_1^{m_1} \cdots p_k^{m_k},\ m_j < 3, \\ 0 & p^3 \mid n \text{ for some prime } p. \end{cases}$$

$r = \infty$: the Liouville function.

$$\mu_\infty(n) = \lambda(n) = (-1)^{m_1+\cdots+m_k} \quad n = p_1^{m_1} \cdots p_k^{m_k}.$$

Definition 3.2. All together, we call $\{\mu_r\}_{r=1}^{\infty}$ the *Möbius functions of higher rank.*

It follows by definition

$$\mu_r(p^m) = \begin{cases} (-1)^m & m < r, \\ 0 & m \geq r \end{cases}$$

for a prime p and $m \geq 1$. Observe that each μ_r is multiplicative; in particular, $|\mu_r|$ is a characteristic function of r-free numbers.

We have already seen that

$$\mu * 1 = u \quad \text{and} \quad \lambda * 1 = \chi_{\mathbf{N}^2}.$$

Now understand this as $\mu_2 * 1 = u$ and $\mu_\infty * 1 = \chi_{\mathbf{N}^2}$. A natural question is: what is $\mu_r * 1$ for $3 \leq r < \infty$? Since μ_r and 1 are both multiplicative, so is $\mu_r * 1$. Now let us see what $(\mu_r * 1)(p^m)$ is.

Proposition 3.3. *Let $r \geq 3$ and $m \geq 1$.*
If $m < r$, then

$$(\mu_r * 1)(p^m) = \begin{cases} 1 & m \text{ even}, \\ 0 & m \text{ odd}. \end{cases}$$

If $m \geq r$, then

$$(\mu_r * 1)(p^m) = \begin{cases} 1 & r \text{ odd}, \\ 0 & r \text{ even}. \end{cases}$$

Proof. Suppose $m < r$. Then

$$\begin{aligned} (\mu_r * 1)(p^m) &= \sum_{d \mid p^m} \mu_r(d) \\ &= \mu_r(1) + \mu_r(p) + \mu_r(p^2) + \cdots + \mu_r(p^m) \\ &= 1 + (-1) + 1 + \cdots + (-1)^m \\ &= \begin{cases} 1 & m \text{ even}, \\ 0 & m \text{ odd}. \end{cases} \end{aligned}$$

If $m \geq r$, then

$$\begin{aligned}
(\mu_r * 1)(p^m) &= \sum_{d \mid p^m} \mu_r(d) \\
&= \mu_r(1) + \mu_r(p) + \mu_r(p^2) + \cdots + \mu_r(p^m) \\
&= \mu_r(1) + \mu_r(p) + \mu_r(p^2) + \cdots + \mu_r(p^{r-1}) + 0 + \cdots + 0 \\
&= 1 + (-1) + 1 + \cdots + (-1)^{r-1} \\
&= \begin{cases} 1 & r \text{ odd}, \\ 0 & r \text{ even}. \end{cases}
\end{aligned}$$

□

Consequently, for $n = p_1^{m_1} \cdots p_k^{m_k}$, the integer

$$(\mu_r * 1)(n) = (\mu_r * 1)(p_1^{m_1} \cdots p_k^{m_k}) = (\mu_r * 1)(p_1^{m_1}) \cdots (\mu_r * 1)(p_k^{m_k})$$

is 1 if and only if all of factors $(\mu_r * 1)(p_j^{m_j})$ are 1. Otherwise, i.e., $(\mu_r * 1)(p_j^{m_j}) = 0$ for some j, it is 0. This naturally leads to an interpretation of $\mu_r * 1$ as a characteristic function of some set as follows. For each $r \geq 3$, define M_r, a subset of $\mathbf{N}$:

- r odd or $r = \infty$: square numbers.

$$M_3 = M_5 = \cdots = M_\infty == \mathbf{N}^2 \, (= \{n \in \mathbf{N} \mid n = p_1^{m_1} \cdots p_k^{m_k}, m_j \text{ all even}\}).$$

- r even: ranked square numbers.

$$M_r = \{n \in \mathbf{N} \mid n = p_1^{m_1} \cdots p_k^{m_k}, m_j \text{ all even}, m_j < r\}.$$

The sets M_r's (r even) are increasing:

$$M_4 \subset M_6 \subset M_8 \subset \cdots \subset M_\infty = \mathbf{N}^2.$$

Example 3.4.

$$\begin{aligned}
M_4 &= \{n \in \mathbf{N} \mid n = p_1^{m_1} \cdots p_k^{m_k}, m_j \text{ all even}, m_j < 4\} \\
&= \left\{n \in \mathbf{N} \,\middle|\, n = 1 \text{ or } n = p_1^2 \cdots p_k^2\right\} \\
&= \{1, 4, 9, 25, 36, 49, 100, 121, 169, \dots\}.
\end{aligned}$$

Proposition 3.5. *Let $M_1 = \mathbf{N}$ and $M_2 = \{1\}$. Then, for each $r \in \mathbf{N} \cup \{\infty\}$, the Dirichlet product $\mu_r * 1$ is a characteristic function of the set M_r:*

$$(\mu_r * 1)(n) = \begin{cases} 1 & n \in M_r, \\ 0 & n \notin M_r. \end{cases}$$

TABLE 4. μ_3 and $\mu_3 * 1$

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\mu_3(n)$	1	-1	-1	1	-1	1	-1	0	1	1	$\cdots$
$(\mu_3 * 1)(n)$	1	0	0	1	0	0	0	0	1	0	$\cdots$

Proposition 3.6 (Möbius functions of higher rank and zeta)**.** *For $r \geq 1$ and $Re(s) > 1$, we have*

$$\left(\sum_{n=1}^{\infty} \frac{\mu_r(n)}{n^s}\right) \zeta(s) = \sum_{n \in M_r} \frac{1}{n^s}.$$

Proof. This statement is equivalent to $\mu_r * 1 = \chi_{M_r}$. □

For clarity, we sometimes prefer to write

$$\sum_{n=1}^{\infty} \frac{\mu_r(n)}{n^s} = \sum_{n:r\text{-free}} \frac{\mu_r(n)}{n^s}.$$

In the next section, we will compute such sums for $s = 2$.

4. Main Theorems

Before going into main theorems, we briefly recall an important family of polynomials in number theory for convenience.

4.1. Cyclotomic polynomials.

The *cyclotomic polynomial* for n is

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} (x - e^{2\pi i n/k}).$$

This is indeed a polynomial of integer coefficients.

Example 4.1.

$$\Phi_1(x) = x - 1, \quad \Phi_2(x) = x + 1, \quad \text{and} \quad \Phi_3(x) = x^2 + x + 1.$$

An important relation to the Möbius function is:

Fact 4.2.

$$\Phi_n(x) = \prod_{d \mid n} (x^d - 1)^{\mu\left(\frac{n}{d}\right)}.$$

Exponents are $0, \pm 1$ so that $\Phi_n(x)$ (and $\Phi_n(x)^{-1}$ also) is a product of $(x^d - 1)$'s. Note that a factor $x^d - 1$ looks like "$1 - p^{-s}$" in the Euler product of $\zeta(s)$; this idea will play a key role in the proofs below.

4.2. **Theorem ($s = 2$).** We are now ready for computing three series in the middle of Table 3.

Theorem 4.3.

(1) $\displaystyle\sum_{n:3\text{-free}} \frac{\mu_3(n)}{n^2} = \frac{45045}{691\pi^4}$.

(2) $\displaystyle\sum_{n:4\text{-free}} \frac{\mu_4(n)}{n^2} = \frac{630}{\pi^6}$.

(3) $\displaystyle\sum_{n:5\text{-free}} \frac{\mu_5(n)}{n^2} = \frac{1091215125}{174611\pi^8}$.

Proof of (1). Note that

$$\begin{aligned}\Phi_{12}(x) &= (x-1)^{\mu(12)}(x^2-1)^{\mu(6)}(x^3-1)^{\mu(4)}(x^4-1)^{\mu(3)}(x^6-1)^{\mu(2)}(x^{12}-1)^{\mu(1)} \\ &= \frac{(1-x^2)(1-x^{12})}{(1-x^4)(1-x^6)} = 1 - x^2 + x^4.\end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{n:3\text{-free}} \frac{\mu_3(n)}{n^2} &= 1 + \sum_{0<m_j<3} \frac{(-1)^{m_1+\cdots+m_k}}{(p_1^{m_1}\cdots p_k^{m_k})^2} \\ &= \prod_p (1 - p^{-2} + p^{-4}) \\ &= \prod_p \frac{(1-p^{-2})(1-p^{-12})}{(1-p^{-4})(1-p^{-6})} \\ &= \frac{\zeta(4)\zeta(6)}{\zeta(2)\zeta(12)} \\ &= \frac{\pi^4}{90}\,\frac{\pi^6}{945}\,\frac{6}{\pi^2}\,\frac{638512875}{691\pi^{12}} \\ &= \frac{45045}{691\pi^4}.\end{aligned}$$

□

Proof of (2). Since

$$1 - x^2 + x^4 - x^6 = \frac{(1-x^2)(1-x^8)}{1-x^4},$$

we have

$$
\begin{aligned}
\sum_{n:4\text{-free}} \frac{\mu_3(n)}{n^2} &= 1 + \sum_{0<m_j<4} \frac{(-1)^{m_1+\cdots+m_k}}{(p_1^{m_1}\cdots p_k^{m_k})^2} \\
&= \prod_p (1 - p^{-2} + p^{-4} - p^{-6}) \\
&= \prod_p \frac{(1-p^{-2})(1-p^{-8})}{1-p^{-4}} \\
&= \frac{\zeta(4)}{\zeta(2)\zeta(8)} \\
&= \frac{\pi^4}{90}\frac{6}{\pi^2}\frac{9450}{\pi^8} \\
&= \frac{630}{\pi^6}.
\end{aligned}
$$

$\square$

Proof of (3). The idea is quite similar. From the cyclotomic polynomial

$$\Phi_{20}(x) = \frac{(1-x^2)(1-x^{20})}{(1-x^4)(1-x^{10})} = 1 - x^2 + x^4 - x^6 + x^8,$$

we obtain

$$
\begin{aligned}
\sum_{n:5\text{-free}} \frac{\mu_5(n)}{n^2} &= 1 + \sum_{0<m_j<5} \frac{(-1)^{m_1+\cdots+m_k}}{(p_1^{m_1}\cdots p_k^{m_k})^2} \\
&= \prod_p (1 - p^{-2} + p^{-4} - p^{-6} + p^{-8}) \\
&= \prod_p \frac{(1-p^{-2})(1-p^{-20})}{(1-p^{-4})(1-p^{-10})} \\
&= \frac{\zeta(4)\zeta(10)}{\zeta(2)\zeta(20)} \\
&= \frac{\pi^4}{90}\frac{\pi^{10}}{93555}\frac{6}{\pi^2}\frac{1531329465290625}{174611\pi^{20}} \\
&= \frac{1091215125}{174611\pi^8}.
\end{aligned}
$$

$\square$

4.3. **Möbius inversion of rank 3.** Recall that $\mu_2^{-1} = 1$. Thus, the equality

$$\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

can be regarded as "Möbius inversion of rank 2". Here we consider the case of rank 3.

For the prime factorization $n = p_1^{m_1} \cdots p_k^{m_k}$ of n into distinct primes, let

$$m_*(n) = |\{j \mid m_j \equiv 3, 4 \bmod 6\}|.$$

Theorem 4.4 (Möbius inversion of rank 3)**.**

$$\left(\sum_{n=1}^{\infty} \frac{\mu_3(n)}{n^s}\right)^{-1} = \sum_{\substack{n=p_1^{m_1}\cdots p_k^{m_k} \\ m_j \not\equiv 2,5 \bmod 6}} \frac{(-1)^{m_*(n)}}{n^s}.$$

Proof. We know that

$$\sum_{n=1}^{\infty} \frac{\mu_3(n)}{n^s} = \sum_{n:3\text{-free}} \frac{\mu_3(n)}{n^s} = \prod_p (1 - (p^{-1})^s + (p^{-2})^s).$$

Now the idea is to find the inverse (formal power series) of $1 - x + x^2$:

$$\begin{aligned}
(1 - x + x^2)^{-1} &= \Phi_6(x)^{-1} \\
&= \frac{(1 - x^2)(1 - x^3)}{(1 - x)(1 - x^6)} \\
&= \frac{1 + x - x^3 - x^4}{1 - x^6} \\
&= \sum_{i=0}^{\infty} x^{6i}(1 + x - x^3 - x^4).
\end{aligned}$$

That is,

$$\sum_{n=1}^{\infty} \frac{\mu_3^{-1}(n)}{n^s} = \prod_p \left(\sum_{i=0}^{\infty} p^{-6si}(1 + p^{-s} - p^{-3s} - p^{-4s})\right).$$

It follows that

$$\sum_{n=1}^{\infty} \frac{\mu_3^{-1}(n)}{n^s} = \prod_p \left(\sum_{i=0}^{\infty} p^{-6si}(1 + p^{-s} - p^{-3s} - p^{-4s})\right) = \sum_{\substack{n=p_1^{m_1}\cdots p_k^{m_k} \\ m_j \not\equiv 2,5 \bmod 6}} \frac{(-1)^{m_*(n)}}{n^s}.$$

$\square$

TABLE 5. $\mu_3(n)$ and $\mu_3^{-1}(n)$

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\mu_3(n)$	1	-1	-1	1	-1	1	-1	0	1	1	$\cdots$
$\mu_3^{-1}(n)$	1	1	1	0	1	1	1	-1	0	1	$\cdots$

Corollary 4.5.

$$\mu_3^{-1}(p^m) = \begin{cases} 1 & m \equiv 0, 1 \mod 6, \\ 0 & m \equiv 2, 5 \mod 6, \\ -1 & m \equiv 3, 4 \mod 6. \end{cases}$$

Corollary 4.6.

$$\sum_{n=1}^{\infty} \frac{\mu_3^{-1}(n)}{n^2} = \frac{691\pi^4}{45045}.$$

Proof. This is the inverse of the sum

$$\left(\sum_{n=1}^{\infty} \frac{\mu_3(n)}{n^2} = \right) \sum_{n:3\text{-free}} \frac{\mu_3(n)}{n^2} = \frac{45045}{691\pi^4}$$

proved in Theorem 4.3. □

Let us put it this way; this gives a new kind of an infinite series expression of π in terms of μ_3^{-1}:

$$\pi = \left(\frac{45045}{691} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{10^2} + \cdots \right) \right)^{1/4}.$$

4.4. **Theorem ($s = 3$).** We evaluated several Dirichlet series at $s = 2$ so that many zeta values at even integers $\{\zeta(2n)\}$ appeared. On the one hand, exact values of $\zeta(2n)$ are known in terms of *Bernoulli numbers*; on the other hand, not much is known on $\{\zeta(2n+1)\}$.

- Apéry [1] proved that $\zeta(3)$ is irrational in 1979.
- More recently, Zudilin [4] proved that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.
- Exact value of any $\zeta(2n+1)$ is not known.

However, our method is helpful for understanding some *relation* of particular series involving $\{\zeta(2n+1)\}$. Let us see an example on $s = 3$, $\zeta(3)$ and $\zeta(9)$ here.

Lemma 4.7.

(1) $\displaystyle\sum_{n:3\text{-free}} \frac{1}{n^3} = \frac{\zeta(3)}{\zeta(9)}.$

(2) $$\sum_{n:\text{3-free}} \frac{\mu_3(n)}{n^3} = \frac{\zeta(6)\zeta(9)}{\zeta(3)\zeta(18)}.$$

Proof. Take the cyclotomic polynomials

$$\Phi_9(x) = \frac{1-x^9}{1-x^3} = 1 + x^3 + x^6 \quad \text{and}$$
$$\Phi_{18}(x) = \frac{(1-x^3)(1-x^{18})}{(1-x^6)(1-x^9)} = 1 - x^3 + x^6.$$

Then

$$\sum_{n:\text{3-free}} \frac{1}{n^3} = \prod_p (1 + (p^{-1})^3 + (p^{-2})^3) = \prod_p \frac{1-p^{-9}}{1-p^{-3}} = \frac{\zeta(3)}{\zeta(9)} \quad \text{and}$$

$$\sum_{n:\text{3-free}} \frac{\mu_3(n)}{n^3} = \prod_p (1 - (p^{-1})^3 + (p^{-2})^3) = \prod_p \frac{(1-p^{-3})(1-p^{-18})}{(1-p^{-3})(1-p^{-6})} = \frac{\zeta(6)\zeta(9)}{\zeta(3)\zeta(18)}.$$

□

Theorem 4.8. $$\left(\sum_{n:\text{3-free}} \frac{1}{n^3}\right)\left(\sum_{n:\text{3-free}} \frac{\mu_3(n)}{n^3}\right) = \frac{41247931725}{43867\pi^{12}}.$$

Proof. Thanks to Lemma 4.7, the left hand side is

$$\frac{\zeta(6)}{\zeta(18)} = \frac{\pi^6}{945} \frac{38979295480125}{43867\pi^{18}} = \frac{41247931725}{43867\pi^{12}}.$$

□

5. Final remarks

5.1. **Lambert series.** Here, we record some of our results in a little different form.

For a sequence a_n of integers, its *Lambert series* is the formal power series

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{1-x^n}.$$

Assume that $a_n = f(n)$ for some arithmetic function f. It turns out that the coefficient of x^N in $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)x^n}{1-x^n}$ is exactly $\displaystyle\sum_{d|N} f(d)$, that is, $(f * 1)(N)$.

Corollary 5.1. *For $r \geq 3$ odd or $r = \infty$, we have*

$$\sum_{n=1}^{\infty} \frac{\mu_r(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} x^{n^2}.$$

In particular, this includes

$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} x^{n^2} = x + x^4 + x^9 + x^{16} + \cdots$$

as a special case.

Corollary 5.2. *For r even, we have*

$$\sum_{n=1}^{\infty} \frac{\mu_r(n)x^n}{1-x^n} = \sum_{\substack{n:n=p_1^{m_1}\cdots p_k^{m_k}\\ m_j<r,\ m_j \text{ even}}} x^n.$$

For example, $r = 4$,

$$\frac{\mu_4(1)x}{1-x} + \frac{\mu_4(2)x^2}{1-x^2} + \frac{\mu_4(3)x^3}{1-x^3} + \frac{\mu_4(4)x^4}{1-x^4} + \frac{\mu_4(5)x^5}{1-x^5} + \cdots$$
$$= x + x^4 + x^9 + x^{25} + x^{36} + x^{49} + x^{100} + x^{121} + x^{169} + \cdots .$$

5.2. **Future research.** We leave several ideas here for our future research.

(1) We expect that there are many more results on Dirichlet series

$$\sum_{n:r\text{-free}} \frac{f(n)}{n^s}, \quad r, s \geq 2, \quad f \in \{\mu_r, \mu_r * 1, \mu_r^{-1}, 1\}.$$

(2) Suppose a multiplicative arithmetic function f satisfies

$$f(p^m) = f(q^m) \quad \text{for all primes } p, q.$$

The *Bell series* for such f is the formal power series

$$B_f(x) = \sum_{m=0}^{\infty} f(p^m)x^m$$

as $B_\mu(x) = 1 - x$, for instance. Say f is *cyclotomic* if $B_f(x) = \Phi_n(x)$ for some n; it is *inverse cyclotomic* if $B_f(x) = \Phi_n(x)^{-1}$ for some n. Study a series $\displaystyle\sum_n \frac{f(n)}{n^s}$ for functions of this class.

(3) Describe details of μ_r^{-1} for $r \geq 4$.

References

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