When does a strict inequality of Kazhdan-Lusztig polynomials hold?

Masato Kobayashi

Kanagawa University

September 19, 2019

Keywords

- **Bruhat graph**
- **Coxeter group**
- **Kazhdan-Lusztig polynomials**

Hisotry: Kazhdan-Lusztig polynomials

Kazhdan-Lusztig introduced this in 1979 to study representation of Hecke algebras and geometry of Schubert varieties.

Since then...

1979 KL polynomials

Geometry and Representation theory:
Braden-MacPherson, Brylinski-Kashiwara, Elias-Williamson, Goresky, Irving, Tanisaki, . . .

Combinatorics:
Billey-Lakshmibai, Björner-Brenti school, Carrell-Peterson, Deodhar, Dyer, . . .

Notation

$(W, S, T, \ell, \leq, \to)$ is a crystallographic Coxeter system:

$$W = S^*/\{(rs)^{m(r,s)} = e, m(s,s) = 1\}$$

$m(r,s) \in \{1, 2, 3, 4, 6, \infty\}$
$T = \{wsw^{-1} \mid w \in W, s \in S\}$,
ℓ: length funciton

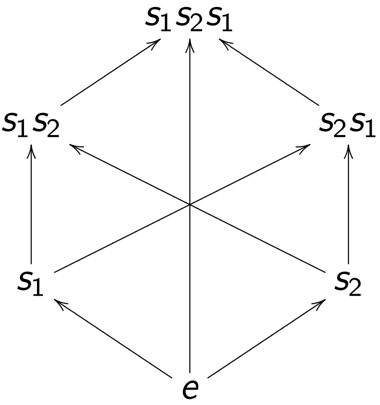
$$\ell(w) = \min\{l \geq 0 \mid w = s_1 \cdots s_l, s_i \in S\}.$$

$\leq, \to$: **Bruhat order, Bruhat graph**

Def: **Bruhat graph** of W is a directed graph for vertices $w \in W$ and for edges $u \to ut$, $t \in T$, $\ell(u) < \ell(ut)$.

$\leq$: **Bruhat order**$=$ transitive closure of this relation.

Example: $W = A_2$



Fact: There exists a unique family of polynomials $\{P_{uw}(q) \mid u, w \in W\} \subseteq \mathbf{Z}[q]$ (**Kazhdan-Lusztig polynomials**) such that

- $P_{uw}(q) = 0$ if $u \not\leq w$,
- $P_{uw}(q) = 1$ if $u = w$,
- $\deg P_{uw}(q) \leq (\ell(u, w) - 1)/2$ if $u < w$,
- $[q^0](P_{uw}) = 1$ if $u \leq w$,
- if $u \leq w$, then

$$q^{\ell(u,w)} P_{uw}(q^{-1}) = \sum_{u \leq v \leq w} \mathbf{R}_{uv}(q) P_{vw}(q).$$

Fact: There exists a unique family of polynomials $\{R_{uw}(q) \mid u, w \in W\} \subseteq \mathbf{Z}[q]$ (R-**polynomials**) such that

- $R_{uw}(q) = 0$ if $u \not\leq w$,
- $R_{uw}(q) = 1$ if $u = w$,
- if $s \in S$ and $\ell(ws) < \ell(w)$, then

$$R_{uw}(q) = \begin{cases} R_{us,ws}(q) & \text{if } \ell(us) < \ell(u), \\ (q-1)R_{u,ws}(q) + qR_{us,ws}(q) & \text{if } \ell(u) < \ell(us). \end{cases}$$

Fact:

$$R_{uw}(1) = \begin{cases} 1 & \text{if } u = w \\ 0 & \text{otherwise.} \end{cases}$$

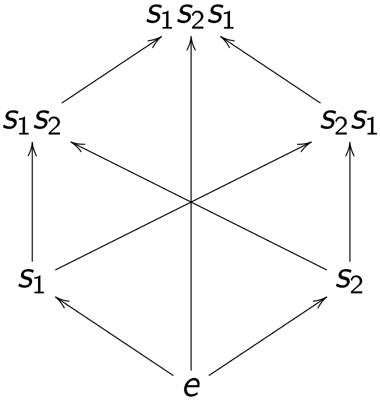
$$R'_{uw}(1) = \begin{cases} 1 & \text{if } u \to w \\ 0 & \text{otherwise.} \end{cases}$$

$\Longrightarrow R_{uw}(q)$ somehow "detects vertices and edges" on Bruhat graphs at $q = 1$.

Relation between Bruhat graph and KL polynomials

Fact (Carrell 1994): For $u \leq w$, the following are equivalent:

- $P_{uw}(1) = 1$.
- $[u, w]$ is regular as a Bruhat graph.



is regular.

$$P_{e,s_1 s_2 s_1}(q) = 1.$$

Fact (Irving 1988, Braden-MacPherson 2001):

- $P_{uw}(q) \geq 0$. (coefficientwise)
- If $u \leq v \leq w$ in W, then

$$P_{uw}(1) \geq P_{vw}(1).$$

However, this would **not** tell us when a strict inequality holds!

Def (Kobayashi): $u \to v$ in $[u, w]$ is **strict** if

$$P_{uw}(1) > P_{vw}(1).$$

Q: how often such inequalities hold in $[u, w]$?

Main theorem

Theorem (Kobayashi 2013): Suppose $u \leq w$. If $P_{uw}(1) > 1$, then there exists $v \in [u, w]$ such that

$$u \to v \quad \text{and} \quad P_{uw}(1) \gneq P_{vw}(1).$$

Proof 1/4

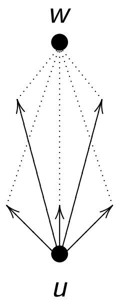
- $\ell(u,w) = \ell(w) - \ell(u)$
- $\overline{\ell}(u,w) = |\{v \in W \mid u \to v \leq w\}|$
- $n \stackrel{\text{def}}{=} |\{v \mid u \to v \leq w, \text{ strict}\}|$

Fact (Deodhar inequality): If $P_{uw}(1) > 1$, then

$$\overline{\ell}(u,w) - \ell(u,w) > 0.$$

Proof 2/4

$$\ell(u, w) = \text{rank of } [u, w] \text{ as an Eulerian poset.}$$
$$\overline{\ell}(u, w) = \text{the number of outgoing edges from } u.$$



Proof 3/4

Goal is to show $n \geq 1$. Suppose $n = 0$.
Differentiate

$$q^{\ell(u,w)} P_{uw}(q^{-1}) = \sum_{u \leq v \leq w} R_{uv}(q) P_{vw}(q)$$

and $q = 1$:

$$\underline{\ell(u,w) P_{uw}(1)} - 2P'_{uw}(1) = \sum_{v : u \to v \leq w} P_{vw}(1)$$

$$= \sum_{\substack{v : u \to v \leq w \\ \text{strict}}} P_{vw}(1) + \sum_{\substack{v : u \to v \leq w \\ \text{not strict}}} P_{vw}(1)$$

Proof 4/4

$$\begin{aligned}\underbrace{-2P'_{uw}(1)}_{<0} &= \sum_{\substack{v: u \to v \leq w \\ \text{strict}}} P_{vw}(1) + (\bar{\ell}(u,w) - \underbrace{n}_{0} - \ell(u,w))P_{uw}(1) \\ &= \underbrace{\sum_{\substack{v: u \to v \leq w \\ \text{strict}}} P_{vw}(1)}_{\geq 0} + \underbrace{(\bar{\ell}(u,w) - \ell(u,w))P_{uw}(1)}_{\geq 0},\end{aligned}$$

a contradiction. □

Further ideas:

- This proof was a brute force. Find a more elegant proof.
- Study these positive integers

$$P_{uw}(1) - P_{vw}(1)$$

  for $u \leq w$.
- Find a combinatorial interpretation of coefficients of Kazhdan-Lusztig polynomials with this theorem.
- Meet Sara Billey in Washington University.

- Kazhdan-Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), no. 2, 165-184.
- Kobayashi, Inequalities on Bruhat graphs, R- and Kazhdan-Lusztig polynomials, J. Comb. Theory Ser. A, 2013, 470-482.

Thanks!