

VALUES OF ZETA-ONE FUNCTIONS AT POSITIVE EVEN INTEGERS

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ABSTRACT. Motivated by Euler-Goldbach and Shallit-Zikan theorems, we introduce zeta-one functions with infinite sums of $n^s \pm 1$ as an analogy of the Riemann zeta function. Then we compute values of these functions at positive even integers by the residue theorem.

CONTENTS

1. Introduction: Euler-Goldbach Theorem	2
2. Zeta-one functions	3
2.1. Definition	3
2.2. Example	3
3. Main theorem 1	4
3.1. Lemmas	4
3.2. Proof of main theorem 1	7
3.3. Example	8
4. Main theorem 2	9
4.1. Lemma	9
4.2. Proof of main theorem 2	11
4.3. Example	11
References	12

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1. INTRODUCTION: EULER-GOLDBACH THEOREM

Let us start with the celebrated *Euler-Goldbach Theorem*. Say that a natural number p is a *perfect power* if $p = n^m$ for some natural numbers $m, n \geq 2$.

Theorem 1.1 (Euler-Goldbach).

$$\sum_{p:\text{perfect power}} \frac{1}{p-1} = 1.$$

See Bibiloni-Paradis-Viader [2] for history of this theorem. Recently, Shallit-Zikan [4] (1983) reinterpreted it in terms of *Riemann's zeta function*: the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

is convergent for all complex numbers s such that $\operatorname{Re}(s) > 1$. Indeed, Euler proved that

$$\zeta(2k) = -\frac{1}{2} \frac{(2\pi i)^{2k}}{(2k)!} B_{2k} \quad (\text{Table 1})$$

where $\{B_n\}$ are signed Bernoulli numbers as in Table 2; refer to Ayoub [1] for history of this function.

Since $\zeta(s) = 1 + \frac{1}{2^s} + \cdots > 1$ and

$$2 > \frac{\pi^2}{6} = \zeta(2) > \zeta(3) > \zeta(4) > \zeta(5) > \cdots,$$

we have $1 < \zeta(s) < 2$ for all $s \geq 2$. That is, $\zeta(s) - 1$ is the *fractional part* of $\zeta(s)$. For example,

$$\begin{aligned} \zeta(2) - 1 &= 0.6449 \dots, \\ \zeta(3) - 1 &= 0.2020 \dots, \\ \zeta(4) - 1 &= 0.0823 \dots, \\ \zeta(5) - 1 &= 0.0369 \dots \end{aligned}$$

Theorem 1.2 (Shallit-Zikan [4]).

$$\sum_{k=2}^{\infty} (\zeta(k) - 1) = 1.$$

Let us now see a similar result.

Theorem 1.3 (See J.M. Borwein-Bradley-Crandall [3, p. 262]).

$$\sum_{k=1}^{\infty} (\zeta(2k) - 1) = \frac{3}{4}.$$

Here we give a proof since it suggests some ideas for our main results.

Proof. Consider the double sequence $a_{nk} = (\frac{1}{n^{2k}})_{n \geq 2, k \geq 1}$ and positive series $\sum_{n \geq 2, k \geq 1} a_{nk}$. We find that

$$\begin{aligned} \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^{2k}} &= \sum_{n=2}^{\infty} \frac{1}{n^2} \sum_{k=0}^{\infty} \left(\frac{1}{n^2}\right)^k \\ &= \sum_{n=2}^{\infty} \frac{1}{n^2} \frac{1}{1 - \frac{1}{n^2}} = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \\ &= \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{3}{4} \end{aligned}$$

so that we can freely switch order of this series. As a consequence,

$$\sum_{k=1}^{\infty} (\zeta(2k) - 1) = \sum_{k=1}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^{2k}} = \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^{2k}} = \frac{3}{4}.$$

□

In this proof, the infinite series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ appeared. It is now natural to think of analogous sums $\sum \frac{1}{n^s \pm 1}$. With this simple idea, this article introduces *zeta-one functions* $\zeta_{+1}(s)$, $\zeta_{-1}(s)$ and we compute values of $\zeta_{+1}(2m)$ and $\zeta_{-1}(2m)$ as main Theorems 3.3 and 4.2.

2. ZETA-ONE FUNCTIONS

2.1. Definition. Throughout k, m, n, N and s each denote a nonnegative integer unless otherwise specified. Further, we assume that $s \geq 2$.

Definition 2.1. Define the *zeta-one functions* by

$$\zeta_{+1}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s + 1} \quad \text{and} \quad \zeta_{-1}(s) = \sum_{n=2}^{\infty} \frac{1}{n^s - 1}$$

(For $s \geq 2$, these sums are indeed convergent as mentioned below). Call each *zeta-plus-one* and *zeta-minus-one* function, respectively.

2.2. Example.

Example 2.2. As seen above, $\zeta_{-1}(2) = \frac{3}{4}$. Moreover, since

$$\coth(\pi z) = \frac{1}{\pi z} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2}, \quad z \in \mathbf{C},$$

the substitution $z = 1$ implies that

$$\zeta_{+1}(2) = \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} = -\frac{1}{2} + \frac{\pi}{2} \coth(\pi).$$

TABLE 1. zeta even values

$2k$	2	4	6	8	10	12	$\dots$
$\zeta(2k)$	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9450}$	$\frac{\pi^{10}}{93555}$	$\frac{691\pi^{12}}{638512875}$	$\dots$

TABLE 2. signed Bernoulli numbers (with $B_3 = B_5 = \dots = 0$)

n	0	1	2	4	6	8
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$
n	10	12	14	16	18	20
B_n	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$

As a consequence,

$$\zeta_{+1}(s) \leq \zeta_{+1}(2) < \infty, \quad \zeta_{-1}(s) \leq \zeta_{-1}(2) < \infty$$

for all $s \geq 2$.

3. MAIN THEOREM 1

Toward the proof of Theorem 3.3 on $\zeta_{+1}(2m)$, we need lemmas.

3.1. **Lemmas.** For $m \geq 1$, set

$$f(z) = \frac{\cot(\pi z)}{z^{2m} + 1} \quad \text{and} \quad \alpha = \exp\left(\frac{\pi i}{2m}\right).$$

Recall that

$$\cot(\pi z) = \frac{1}{\pi z} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}, \quad z \in \mathbf{C}.$$

Thus, f is a meromorphic function with simple poles $z = 0, \pm 1, \pm 2, \pm 3, \dots$ and

$$\alpha, \alpha^3, \alpha^5, \dots, \alpha^{2m-1}, -\alpha, -\alpha^3, -\alpha^5, \dots, -\alpha^{2m-1}$$

as all the roots of $z^{2m} + 1 = 0$. Let us compute the residue of f at each pole.

Lemma 3.1. For $n = 0, \pm 1, \pm 2, \pm 3, \dots$, we have

$$\text{Res}(f, n) = \frac{1}{\pi(n^{2m} + 1)},$$

and for $1 \leq k \leq m$,

$$\text{Res}(f, \pm \alpha^{2k-1}) = -\frac{\alpha^{2k-1} \cot(\pi \alpha^{2k-1})}{2m}.$$

Proof. First, we have

$$\text{Res}(f, n) = \lim_{z \rightarrow n} (z - n) f(z) = \lim_{z \rightarrow n} \frac{z - n}{\sin \pi(z - n)} \cdot \frac{\cos(\pi z)}{z^{2m} + 1} = \frac{1}{\pi(n^{2m} + 1)}.$$

Second, for $1 \leq k \leq m$,

$$\begin{aligned} \text{Res}(f, \alpha^{2k-1}) &= \lim_{z \rightarrow \alpha^{2k-1}} (z - \alpha^{2k-1}) f(z) \\ &= \lim_{z \rightarrow \alpha^{2k-1}} (z - \alpha^{2k-1}) \frac{\cot(\pi z)}{z^{2m} + 1} \\ &= \lim_{z \rightarrow \alpha^{2k-1}} \cot(\pi z) \lim_{z \rightarrow \alpha^{2k-1}} \frac{z - \alpha^{2k-1}}{z^{2m} + 1} \\ &= \cot(\pi \alpha^{2k-1}) \lim_{z \rightarrow \alpha^{2k-1}} \frac{1}{2m z^{2m-1}} \quad (\text{L'Hôpital's rule}) \\ &= \cot(\pi \alpha^{2k-1}) \frac{1}{2m(\alpha^{2k-1})^{2m-1}} \\ &= -\frac{\alpha^{2k-1} \cot(\pi \alpha^{2k-1})}{2m} \quad ((\alpha^{2k-1})^{2m} = -1). \end{aligned}$$

It is quite similar to show that

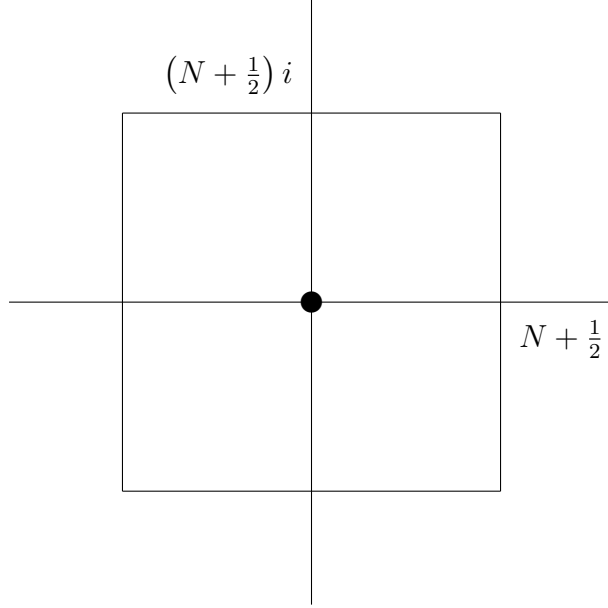
$$\text{Res}(f, -\alpha^{2k-1}) = -\frac{\alpha^{2k-1} \cot(\pi \alpha^{2k-1})}{2m}.$$

□

Lemma 3.2. For a positive integer N , consider line segments on the complex plane

$$\begin{aligned} C_1(N) &= \left\{ \left(N + \frac{1}{2} \right) + yi \mid - \left(N + \frac{1}{2} \right) \leq y \leq N + \frac{1}{2} \right\}, \\ C_2(N) &= \left\{ x + \left(N + \frac{1}{2} \right) i \mid - \left(N + \frac{1}{2} \right) \leq x \leq N + \frac{1}{2} \right\}, \\ C_3(N) &= \left\{ - \left(N + \frac{1}{2} \right) + yi \mid - \left(N + \frac{1}{2} \right) \leq y \leq N + \frac{1}{2} \right\}, \\ C_4(N) &= \left\{ x - \left(N + \frac{1}{2} \right) i \mid - \left(N + \frac{1}{2} \right) \leq x \leq N + \frac{1}{2} \right\} \end{aligned}$$

and set $C(N) = C_1(N) \cup C_2(N) \cup C_3(N) \cup C_4(N)$.



[1] If $z \in C(N)$, then $|\cot(\pi z)| \leq \coth \frac{3}{2}\pi$.

[2] If $z \in C(N)$, then

$$\frac{1}{|z^{2m} + 1|} \leq \frac{1}{(N + \frac{1}{2})^{2m} - 1}.$$

Proof.

[1] Suppose $z \in C(N)$. If $z \in C_1(N)$, then write

$$z = \left(N + \frac{1}{2}\right) + yi, \quad -\left(N + \frac{1}{2}\right) \leq y \leq N + \frac{1}{2}.$$

$$\begin{aligned} |\cot \pi z| &= \left| \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} \right| = \left| \frac{e^{-\pi y} e^{(N+\frac{1}{2})\pi i} + e^{\pi y} e^{-(N+\frac{1}{2})\pi i}}{e^{-\pi y} e^{(N+\frac{1}{2})\pi i} - e^{\pi y} e^{-(N+\frac{1}{2})\pi i}} \right| \\ &= \left| \frac{e^{-\pi y}(-1)^N i + e^{\pi y}(-1)^N (-i)}{e^{-\pi y}(-1)^N i - e^{\pi y}(-1)^N (-i)} \right| \\ &= \left| \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}} \right| = \left| \frac{e^{\pi y} - e^{-\pi y}}{e^{\pi y} + e^{-\pi y}} \right| \\ &= |\tanh(y)| \leq 1 < \coth \frac{3}{2}\pi (= 1.00016 \dots). \end{aligned}$$

If $z \in C_2(N)$, then

$$z = x + \left(N + \frac{1}{2}\right)i, \quad -\left(N + \frac{1}{2}\right) \leq x \leq N + \frac{1}{2}$$

and

$$\begin{aligned}
|\cot(\pi z)| &= \left| \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} \right| \\
&\leq \frac{|e^{\pi iz}| + |e^{-\pi iz}|}{||e^{\pi iz}| - |e^{-\pi iz}||} \\
&= \frac{|e^{-\pi(N+\frac{1}{2})}| + |e^{\pi(N+\frac{1}{2})}|}{||e^{-\pi(N+\frac{1}{2})}| - |e^{\pi(N+\frac{1}{2})}||} \\
&= \frac{e^{\pi(N+\frac{1}{2})} + e^{-\pi(N+\frac{1}{2})}}{e^{\pi(N+\frac{1}{2})} - e^{-\pi(N+\frac{1}{2})}} \\
&= \coth\left(N + \frac{1}{2}\right) \pi \leq \coth \frac{3}{2} \pi
\end{aligned}$$

since $t \mapsto \coth(t)$ is decreasing for $t > 0$. For $z \in C_3(N) \cup C_4(N)$, we have $-z \in C_1(N) \cup C_2(N)$ so that

$$|\cot(\pi z)| = |-\cot(\pi(-z))| = |\cot(\pi(-z))| \leq \coth \frac{3}{2} \pi.$$

[2] If $z \in C(N)$, then $|z| \geq N + \frac{1}{2}$. Consequently, $|z|^{2m} - 1 \geq (N + \frac{1}{2})^{2m} - 1$,

$$\begin{aligned}
\frac{1}{|z|^{2m} - 1} &\leq \frac{1}{(N + \frac{1}{2})^{2m} - 1}, \\
\frac{1}{|z|^{2m} + 1} &\leq \frac{1}{|z|^{2m} - 1} \leq \frac{1}{(N + \frac{1}{2})^{2m} - 1}.
\end{aligned}$$

□

3.2. Proof of main theorem 1. Let s be a positive even integer, say $s = 2m$, $m \geq 1$. Further, let $\alpha = \alpha_{2m} = \exp\left(\frac{\pi i}{2m}\right)$ for convenience.

Theorem 3.3.

$$\zeta_{+1}(2m) = -\frac{1}{2} + \frac{1}{2m} \sum_{k=1}^m \pi \alpha^{2k-1} \cot(\pi \alpha^{2k-1}).$$

Proof. View $C(N) = C_1(N) + C_2(N) + C_3(N) + C_4(N)$ above as the sum of four paths with counterclockwise orientation. Notice that any pole of f does not lie on $C(N)$. We are going to compute the integral

$$I_N = \int_{C(N)} f(z) dz.$$

Let $D(N)$ be the domain enclosed by $C(N)$. Then, the residue theorem with Lemma 3.1 claims that

$$\begin{aligned}
I_N &= \int_{C(N)} f(z) dz \\
&= 2\pi i \sum_{\substack{a: \text{pole of } f(z) \\ a \in D(N)}} \text{Res}(f, a) \\
&= 2\pi i \left(\text{Res}(f, 0) + \sum_{n=1}^N (\text{Res}(f, n) + \text{Res}(f, -n)) + \sum_{k=1}^m (\text{Res}(f, \alpha^{2k-1}) + \text{Res}(f, -\alpha^{2k-1})) \right) \\
&= 2\pi i \left(\frac{1}{\pi} + 2 \sum_{n=1}^N \frac{1}{\pi(n^{2m} + 1)} + 2 \left(- \sum_{k=1}^m \frac{\alpha^{2k-1} \cot(\pi \alpha^{2k-1})}{2m} \right) \right)
\end{aligned}$$

while Lemma 3.2 implies

$$\begin{aligned}
|I_N| &= \left| \int_{C(N)} f(z) dz \right| \\
&\leq \int_{C(N)} |f(z)| dz \\
&\leq \frac{\coth \frac{3}{2}\pi}{(N + \frac{1}{2})^{2m} - 1} \int_{C(N)} dz \\
&= \frac{\coth \frac{3}{2}\pi}{(N + \frac{1}{2})^{2m} - 1} \cdot 8 \left(N + \frac{1}{2} \right) \rightarrow 0 \quad (N \rightarrow \infty).
\end{aligned}$$

Therefore, taking the limit $N \rightarrow \infty$ for I_N yields

$$0 = 2\pi i \left(\frac{1}{\pi} + \frac{2}{\pi} \zeta_{+1}(2m) - 2 \sum_{k=1}^m \frac{\alpha^{2k-1} \cot(\pi \alpha^{2k-1})}{2m} \right).$$

Conclude that

$$\zeta_{+1}(2m) = -\frac{1}{2} + \frac{1}{2m} \sum_{k=1}^m \pi \alpha^{2k-1} \cot(\pi \alpha^{2k-1}).$$

□

3.3. Example. Of course, $\zeta_{+1}(2m)$ is a real number so that there should be some expression of $\zeta_{+1}(2m)$ in terms of only real trigonometric functions.

Example 3.4. Let $s = 4, m = 2$ and $\alpha = \alpha_4 = \exp\left(\frac{\pi i}{4}\right)$. Then

$$\begin{aligned}\zeta_{+1}(4) &= \sum_{n=1}^{\infty} \frac{1}{n^4 + 1} = -\frac{1}{2} + \frac{\pi}{4} \sum_{k=1}^2 \alpha \cot(\pi \alpha) \\ &= -\frac{1}{2} + \frac{\pi}{4} (\alpha \cot(\pi \alpha) + \alpha^3 \cot(\pi \alpha^3)) \\ &= -\frac{1}{2} + \frac{\pi}{4} (\alpha \cot(\pi \alpha) + \alpha^{-1} \cot(\pi \alpha^{-1})).\end{aligned}$$

Now, it follows from the facts

$$\pi \exp\left(\pm \frac{\pi i}{4}\right) = \frac{1 \pm i}{\sqrt{2}} \pi \quad \text{and} \quad \cot(x + yi) = \frac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x} \quad x, y \in \mathbf{R}$$

that

$$\begin{aligned}\zeta_{+1}(4) &= -\frac{1}{2} + \frac{\pi}{4} (\alpha \cot(\pi \alpha) + \alpha^{-1} \cot(\pi \alpha^{-1})) \\ &= -\frac{1}{2} + \frac{\pi}{4} \left(\frac{1+i}{\sqrt{2}} \cdot \frac{\sin \sqrt{2}\pi - i \sinh \sqrt{2}\pi}{\cosh \sqrt{2}\pi - \cos \sqrt{2}\pi} + \frac{1-i}{\sqrt{2}} \cdot \frac{\sin \sqrt{2}\pi + i \sinh \sqrt{2}\pi}{\cosh \sqrt{2}\pi - \cos \sqrt{2}\pi} \right) \\ &= -\frac{1}{2} + \frac{\sqrt{2}\pi}{4} \left(\frac{\sin \sqrt{2}\pi + \sinh \sqrt{2}\pi}{\cosh \sqrt{2}\pi - \cos \sqrt{2}\pi} \right).\end{aligned}$$

Let $s = 6, m = 3$ and $\alpha = \alpha_6 = \exp\left(\frac{\pi i}{6}\right) = \frac{\sqrt{3}+i}{2}$. With $\alpha^3 = i, \alpha^5 = \alpha^{-1} = \frac{\sqrt{3}-i}{2}$ and $i \cot(\pi i) = \coth(\pi)$, we observe that

$$\begin{aligned}\zeta_{+1}(6) &= -\frac{1}{2} + \frac{\pi}{6} (\alpha \cot(\pi \alpha) + \alpha^3 \cot(\pi \alpha^3) + \alpha^5 \cot(\pi \alpha^5)) \\ &= -\frac{1}{2} + \frac{\pi}{6} (\alpha \cot(\pi \alpha) + \alpha^{-1} \cot(\pi \alpha^{-1}) + i \cot(\pi i)) \\ &= -\frac{1}{2} + \frac{\pi}{6} \left(\frac{\sqrt{3}+i}{2} \cdot \frac{\sin \sqrt{3}\pi - i \sinh \pi}{\cosh \pi - \cos \sqrt{3}\pi} + \frac{\sqrt{3}-i}{2} \cdot \frac{\sin \sqrt{3}\pi + i \sinh \pi}{\cosh \pi - \cos \sqrt{3}\pi} + \coth(\pi) \right) \\ &= -\frac{1}{2} + \frac{\pi}{6} \left(\frac{\sqrt{3} \sin \sqrt{3}\pi + \sinh \pi}{\cosh \pi - \cos \sqrt{3}\pi} + \coth(\pi) \right).\end{aligned}$$

4. MAIN THEOREM 2

Next, we prove Theorem 4.2 on $\zeta_{-1}(2m)$. Ideas are quite same.

4.1. **Lemma.** For $m \geq 1$, let

$$g(z) = \frac{\cot(\pi z)}{z^{2m} - 1} \quad \text{and} \quad \beta = \exp\left(\frac{\pi i}{m}\right).$$

It has poles at $z = 0, \pm 1, \pm 2, \pm 3, \dots$, and $z = \beta^k$, $1 \leq k \leq 2m-1, k \neq m$. The order of the poles $z = \pm 1$ is 2 and the all others are simple.

Lemma 4.1. For $n = 0, \pm 2, \pm 3, \dots$, we have

$$\operatorname{Res}(g, n) = \frac{1}{\pi(n^{2m} - 1)},$$

for $1 \leq k \leq 2m-1, k \neq m$,

$$\operatorname{Res}(\pi, \beta^k) = \frac{\beta^k}{2m} \cot(\pi\beta^k)$$

and moreover

$$\operatorname{Res}(g, \pm 1) = -\frac{2m-1}{4m\pi}.$$

Proof. The proofs of the first two equalities are almost similar to ones for Lemma 3.1. Thus we only need to show $\operatorname{Res}(g, \pm 1) = -\frac{2m-1}{4m\pi}$.

Let $\phi(z) = \sum_{k=0}^{2m-1} z^k$. Notice that $\phi(z) = (z^{2m+1} - 1)/(z - 1)$.

Then

$$\begin{aligned} \operatorname{Res}(g, 1) &= \lim_{z \rightarrow 1} \frac{d}{dz} (z-1)^2 g(z) \\ &= \lim_{z \rightarrow 1} \frac{d}{dz} (z-1) \cot(\pi z) \cdot \frac{1}{\phi(z)} \\ &= \lim_{z \rightarrow 1} \left(\frac{\cot(\pi z) - \pi(z-1)(\cot^2 \pi z + 1)}{\phi(z)} - (z-1) \cot(\pi z) \frac{\phi'(z)}{\phi(z)^2} \right). \end{aligned}$$

Let us see the first term. Immediately, $\lim_{z \rightarrow 1} \phi(1) = 2m$ and

$$\begin{aligned} \lim_{z \rightarrow 1} (\cot(\pi z) - \pi(z-1)(\cot^2 \pi z + 1)) &\stackrel{w=z-1}{=} \lim_{w \rightarrow 0} (\cot(\pi w) - \pi w(\cot^2 \pi w + 1)) \\ &= \lim_{w \rightarrow 0} (\cot(\pi w)(1 - \pi w \cot(\pi w))) - \lim_{w \rightarrow 0} \pi w \\ &= \lim_{w \rightarrow 0} \frac{\tan(\pi w) - \pi w}{\tan^2 \pi w} - 0 \\ &= \lim_{w \rightarrow 0} \frac{\pi(1 + \tan^2 \pi w) - \pi}{2\pi \tan \pi w(1 + \tan^2 \pi w)} = 0. \quad (\text{L'Hôpital's rule}) \end{aligned}$$

In addition, since

$$\phi'(1) = \sum_{k=0}^{2m-1} k = m(2m-1),$$

the limit $z \rightarrow 1$ for the second term is

$$\begin{aligned} \lim_{z \rightarrow 1} \left(-(z-1) \cot(\pi z) \frac{\phi'(z)}{\phi(z)^2} \right) &= -\lim_{w \rightarrow 0} w \cot(\pi w) \frac{\phi'(w-1)}{\phi(w-1)^2} \\ &= -\frac{1}{\pi} \frac{m(2m-1)}{(2m)^2} = -\frac{2m-1}{4m\pi}. \end{aligned}$$

□

4.2. Proof of main theorem 2. Let $\beta = \exp\left(\frac{\pi i}{m}\right)$ as above.

Theorem 4.2.

$$\zeta_{-1}(2m) = \frac{1}{2} + \frac{2m-1}{4m} - \frac{\pi}{4m} \sum_{1 \leq k \leq 2m-1, k \neq m} \beta^k \cot(\pi \beta^k).$$

Proof. Let $N, C(N), D(N)$ be as in the previous section. Again, the residue theorem claims that

$$\int_{C(N)} g(z) dz = 2\pi i \sum_{\substack{a: \text{pole of } g \\ a \in D(N)}} \text{Res}(g, a).$$

Taking the limit $N \rightarrow \infty$, the integral converges to 0 likewise. It follows from Lemma 4.1 that

$$\begin{aligned} 0 &= 2\pi i \sum_{a: \text{pole of } g} \text{Res}(g, a), \\ 0 &= \text{Res}(g, 0) + \text{Res}(g, 1) + \text{Res}(g, -1) + \sum_{n=2}^{\infty} (\text{Res}(g, n) + \text{Res}(g, -n)) \\ &+ \sum_{1 \leq k \leq 2m-1, k \neq m} \frac{\pi}{2m} \beta^k \cot(\pi \beta^k) \\ &= -\frac{1}{\pi} - \frac{2m-1}{2m} \pi + \frac{2}{\pi} \zeta_{-1}(2m) + \sum_{1 \leq k \leq 2m-1, k \neq m} \frac{\pi}{2m} \beta^k \cot(\pi \beta^k). \end{aligned}$$

Conclude that

$$\zeta_{-1}(2m) = \frac{1}{2} + \frac{2m-1}{4m} - \frac{\pi}{4m} \sum_{1 \leq k \leq 2m-1, k \neq m} \beta^k \cot(\pi \beta^k).$$

□

4.3. Example.

Example 4.3. For $s = 4, m = 2, \beta = \exp\left(\frac{\pi i}{2}\right) = i$, an expression of real trigonometric function for $\zeta_{-1}(4)$ is

$$\zeta_{-1}(4) = \frac{1}{2} + \frac{3}{8} - \frac{\pi}{8} (i \cot(\pi i) + i^3 \cot(\pi i^3)) = \frac{7}{8} - \frac{\pi}{4} \coth(\pi).$$

Notice that this also shows that

$$\sum_{k=1}^{\infty} (\zeta(4k) - 1) = \frac{7}{8} - \frac{\pi}{4} \coth(\pi)$$

as in [3, p.263]. For $s = 6, m = 3, \beta = \exp\left(\frac{\pi i}{3}\right)$, we see that

$$\begin{aligned}
 \zeta_{-1}(6) &= \frac{1}{2} + \frac{5}{12} \\
 -\frac{\pi}{12} (e^{\pi i/3} \cot(\pi e^{\pi i/3}) + e^{-\pi i/3} \cot(\pi e^{-\pi i/3}) + e^{2\pi i/3} \cot(\pi e^{2\pi i/3}) + e^{-2\pi i/3} \cot(\pi e^{-2\pi i/3})) \\
 &= \frac{11}{12} - \frac{\pi}{12} \left(\frac{\sqrt{3} \sinh \sqrt{3}\pi}{\cosh \sqrt{3}\pi + 1} + \frac{\sqrt{3} \sinh \sqrt{3}\pi}{\cosh \sqrt{3}\pi + 1} \right) \\
 &= \frac{11}{12} - \frac{\pi}{12} \left(2\sqrt{3} \frac{2 \sinh \frac{\sqrt{3}}{2}\pi \cosh \frac{\sqrt{3}}{2}\pi}{2 \cosh^2 \frac{\sqrt{3}}{2}\pi} \right) \\
 &= \frac{11}{12} - \frac{\sqrt{3}}{6} \pi \tanh \frac{\sqrt{3}}{2} \pi.
 \end{aligned}$$

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