An extension of qsym functions by Gessel and Assaf-Searles

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Workshop

Representation theory, Algebraic combinatorics, and related topics

2026 May-June, at Kyoto RIMS.

You are always welcome.

Part 1 with Matsumura, Sugimoto

- little introduction
- report our main theorem

Part 2 on my own

- recent progress
- suggest some ideas for future work

Kewords

Part 1 [type C]

- quasi-sym. function
- semistandard oscillating tableaux
- Gessel, Assaf-Searles' theorem

Part 2 [type A]

- F-, Schur-positivity
- quasi-crystal
- crystal skeleton

Little history

- Gessel 1984
  decomposition of s_λ into qsym functions
- Assaf-Searles 2017
  improvement of Gessel's idea
- Choi-Kim-Lee 2024
  introduction of **semistandard oscillating tableaux.**

$x = (x_1, x_2, \ldots)$: variables.

Def

Let $a = (a_1, \ldots, a_m) \in \mathbf{N}^m$. The **monomial qsym function** for a is

$$M_a(x) = \sum x_{i_1}^{a_1} \cdots x_{i_m}^{a_m}$$

with all $(i_1, \ldots, i_m)$ such that $i_1 < \cdots < i_m$.

For example,

$$M_{74}(x) = x_1^7 x_2^4 + x_1^7 x_3^4 + \cdots + x_2^7 x_3^4 + \cdots .$$

Below, a, b are weak compositions, λ is always a partition.

Def

b refines a if we can obtain b from a by repeating " split one entry to two adjacent positive entries".

For example, $a = (\mathbf{7}, \mathbf{4})$

$$(\mathbf{1}, \mathbf{6}, \mathbf{4}), \quad (\mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{2})$$

are refinements of a.

Def

Fundamental qsym function for a is

$$F_a(\boldsymbol{x}) = \sum_{b \in \mathrm{ref}(a)} M_b(\boldsymbol{x})$$

where $\mathrm{ref}(a)$ = the set of all refinements of a.

Example.

$$F_{22} = M_{22} + M_{211} + M_{112} + M_{1111}$$

Let T be a standard tableaux of shape λ. The **descent** set of T is

$$\mathrm{Des}(T) = \{i \mid i+1 \text{ is below } i \text{ in } T\}$$

If

$$\mathrm{Des}(T) = (a_1, \ldots, a_m),$$

then

$$\mathrm{des}(T) = (a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_m).$$

Standardization of an SSYT T:
Give numbers all 1's in T from left.
Next, give numbers all 2's in T from left.
and so on. Finally, we get an ST, T^{st}.

ex

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 8 \\ \hline 2 & 7 \\ \cline{1-2}\end{array} \qquad T^{\mathrm{st}} = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

$$T_1 \sim T_2 \iff T_1^{\mathrm{st}} = T_2^{\mathrm{st}}$$

is an equivalent class. In particular, standardization preserves descent.

Thm (Gessel, 1984)

$$s_\lambda(\boldsymbol{x}) = \sum_{T \in \mathrm{ST}(\lambda)} F_{\mathrm{des}\,T}(\boldsymbol{x})$$

where $\mathrm{ST}(\lambda)$ is the set of all STs of shape λ.

Decompose a symmetric function to qsyms!

Our main theorem: type-A-to-C-extension of Gessel's theorem

type C tableaux?

P-side

- *King tableaux*
- *Kashiwara-Nakashima tableaux*

Q-side

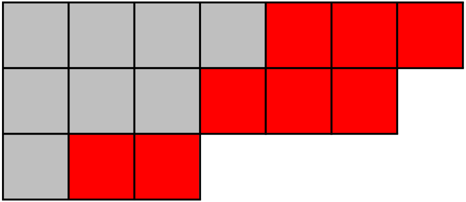
- ***OT***
- ***SSOT***

Def

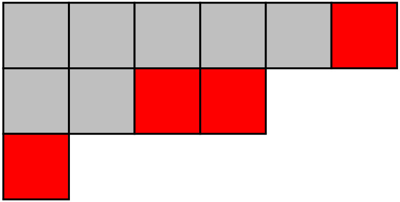
An **oscillating tableau** (OT) is a sequence of partitions with starting at $\emptyset$, and at each step, we add or delete one box.

$$\left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{1,1}\right)$$

A skew shape is a difference of two Young diagrams.



A skew shape is **a horizontal strip** if it contains at most one box in each row.



Def

Let $O = (O_j)$ be an OT of length n. Define $d(O)$ as the sequence

$$(1 \leq d_1 \leq d_{2'} \leq d_2 \leq d_{3'} \leq \cdots \leq d_{k-1} \leq n-1)$$

where $d_{1'} = 0$ with:

- $O_{d_{i'}} \lhd O_{d_{i'}+1} \lhd \cdots \lhd O_{d_i}$, $O_{d_i}/O_{d_{i'}}$ is a horizontal strip and $O_{d_i} \rhd O_{d_i+1}$ or $O_{d_i} \lhd O_{d_i+1}$ but $O_{d_i+1}/O_{d_{i'}}$ is not a horizontal strip.
- $O_{d_i} \rhd O_{d_i+1} \rhd \cdots \rhd O_{d_{(i+1)'}}$ and $O_{d_i}/O_{d_{(i+1)'}}$ is a horizontal strip. $O_{d_{(i+1)'}} \lhd O_{d_{(i+1)'}+1}$ or $O_{d_{(i+1)'}} \rhd O_{d_{(i+1)'}+1}$ but $O_{d_i}/O_{d_{(i+1)'}}$ is not.

Now define **its descent**

$$\operatorname{des} O = (d_i - d_{i-1})_{1 \leq i \leq k}$$

Def (Choi-Kim-Lee)

A semistandard oscillating tableaux (SSOT) of shape λ is a sequence

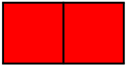
$$S = (S^1, S'^2, S^2, S'^3, \dots, S'^k, S^k)$$

of partitions such that:

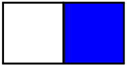
- $S^i \supseteq S'^{i+1}$ and $S'^i \subseteq S^i$. These are horizontal strips.
- $S^k = \lambda$.

Denote by $\mathrm{SSOT}_k(\lambda)$ the set of all such.

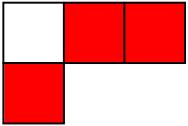
step 1 addition



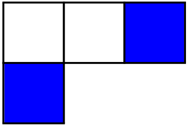
step 2 deletion



step 2 addition



step 3 deletion



step 1 addition

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$$

step 2 deletion

$$\begin{array}{|c|c|}\hline & 2 \\ \hline \end{array}$$

step 2 addition

$$\begin{array}{|c|c|c|}\hline & 2 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}$$

step 3 deletion

$$\begin{array}{|c|c|c|}\hline & & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$$

Identify this

$$S = \begin{array}{|c|c|c|}\hline 1 & 122 & 23 \\ \hline 23 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

standardizaiton of SSOT

step 1: Give numbers to added boxes from left.

step 2 deletion: Give numbers to deleted boxes from right.

step 2 addition: Give numbers to added boxes from left.

and so on.

$$S = \begin{array}{|c|c|c|}\hline 1 & 122 & 23 \\ \hline 23 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \qquad S^{\text{st}} = \begin{array}{|c|c|c|}\hline 1 & 246 & 38 \\ \hline 57 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

Def

$$S_1 \sim S_2 \iff S_1^{\text{st}} = S_2^{\text{st}}$$

This is an equivalent relation on all SSOTs. This preserves des.

weight monomial

$$x^S = \prod_{x_i \in S} x_i.$$

Def

SSOT function

$$ss_\lambda(x) = \sum_S x^S.$$

The sum ranges over all SSOTs of shape λ.

Main Thm 1 (2025+)

$$ss_\lambda(\boldsymbol{x}) = \sum_{T \in \mathrm{OT}(\lambda)} F_{\mathrm{des}T}(\boldsymbol{x}).$$

In particular, this is F-positive.

Idea of proof.

Sum up all weights for standard equivalent classes with each representative $T \in OT(\lambda)$.

Rmk

representation-theoretic approaches on this topic.

- Sundaram
- Naito-Sagaki
- Rubey-Sagan-Westbury
- Heo-Kwon

Main Thm 2

$$ss_\lambda(x) = \sum_{\beta' even} c^\nu_{\beta\lambda} s_\nu(x).$$

In particular, this is Schur positive and hence **symmetric**.

Rmk

Watanabe pointed out that we can interpret our results in terms of reps. of A_{II} quantum symmetric pair.

Part 2

Back to type A.

$$X = (x_1, x_2, \ldots)$$

Classical Problem

Let $g(X) \in \mathbf{C}[[X]]$ be **symmetric** and F-positive. Find partitions μ and $c_\mu \in \mathbf{N}$ such that

$$g(X) = \sum_\mu c_\mu s_\mu.$$

Simpler problem

When is an F-positive function $g(X)$ a **single** Schur?

2010's: Assaf, Roberts, Egge-Loehr-Warrington gave a partial answer.

2023-: Maas-Gariépy, Brauner-Corteel-Daugherty-Schilling started the brand new research with **crystals**.

- quasi-crystal
- crystal skeleton

Consider a **crystal graph** on

$$\mathbf{B}(\lambda) = (\mathsf{SSYT}(\lambda), e_i, f_i, \to).$$

Fact

$\mathbf{B}(\lambda)$ is connected.

Def (Maas-Gariépy)

A **quasi-crystal** is an equivalent class of standardization in $\mathbf{B}(\lambda)$.

For now, this is just a set. However, there is a reason to call it a quasi-**crystal**.

Fact

For each $T \in ST(\lambda)$, a quasi-crystal $[T]$ forms a connected subgraph in $\mathbf{B}(\lambda)$.

Example

$\mathbf{B}_3\left(\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array}\right)$ splits into two quasi-crystals.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Each edge comes from some f_i.

Thm

The generating function of a **quasi**-crystal is a fundamental **quasi**-sym function.

Consequently, $\mathbf{B}(\lambda)$ splits into a union of quasi-crystals.

This gives a deeper understanding for

$$s_\lambda = \sum_{T \in ST(\lambda)} F_{\mathrm{des}(T)}.$$

LR rule revisited

It is thus possible to expand everything

$$s_\lambda s_\mu = \sum c^\nu_{\lambda\mu} s_\nu$$

into F_α's.

What then can we say about $F_\alpha F_\beta$? It is indeed F-**positive**.

Thm (Assaf-Searles 2017)

For all strong compositions α, β,

$$F_\alpha F_\beta = \sum C_{\alpha\beta}^{\gamma} F_\gamma, \quad \exists C_{\alpha\beta}^{\gamma} \in \mathbf{Z}_{\geq 0}.$$

Rmk

This is a **shape-free** discussion!

That is, possibly ∃ distinct STs of same shape with same descent such as

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}\ , \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}\ .$$

Thus,

$$F_\alpha^\lambda = \sum_{T \in ST(\lambda), \mathrm{des}T = \alpha} F_\alpha$$

is the appropriate **shape-dependent** grouping in a single $\mathbf{B}(\lambda)$.

Write $F_\alpha^\lambda = f_{\lambda\alpha} F_\alpha$ with $f_{\lambda\alpha} \in \mathbf{Z}_{\geq 0}$.

Ob (quasi LR)

$$s_\lambda s_\mu = \sum \left(c_{\lambda\mu}^{\nu} f_{\nu\gamma} \right) F_\gamma$$

This is the refinement of LR rule in this context.

Next direction

Understand numbers $(f_{\lambda\alpha})$.

This is **quasi-Kostka numbers**. Only few authors have mentioned this before.

Def (Maas-Gariépy)

A **crystal skeleton** $ST(\lambda)$ for λ is a quotient set $\mathbf{B}(\lambda)/\sim_{st}$.

Def (K.)

A **skeleton polynomial** for λ:

$$\mathrm{Sk}_\lambda(x) = \sum_\alpha f_{\lambda\alpha} x^\alpha.$$

(Think this as generating function of $ST(\lambda)$ wrt. des)

cf.

$$s_\lambda(X) = \sum_\alpha f_{\lambda\alpha} F_\alpha(X).$$

example

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5 \\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4 \\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4 \\ \cline{1-2}\end{array}$$

These 5 STs correspond to

$$\mathrm{Sk}_{\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}}(x) = x^{32} + x^{221} + x^{131} + x^{122} + x^{23}$$

where

$$x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

$\mathrm{Sk}_\lambda(x)$ is not really symmetric, but **“close to"**.

For last two months, I studied those polynomials:

- inner crystal symmetry
- skeleton RS correspondence
- Lusztig symmetry
- Young Branching rule?

Let

$$l = \min\{\ell(\mathrm{des}T) \mid T \in ST(\lambda)\}.$$

Thm (BCDS 2025)

A crystal skeleton graph $ST(\lambda)$ (detail omitted) contains $\mathbf{B}_l(\lambda)$ as a subgraph. Call this the **inner crystal** of $\mathbf{B}(\lambda)$.

$$\mathbf{B}_l(\lambda) \subseteq ST(\lambda) \subseteq \mathbf{B}(\lambda)$$

$\mathbf{B}_3\left(\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}\right)$ consists of two quasi-crystals.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow {\color{red}\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} \longrightarrow \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \longrightarrow {\color{red}\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} \longrightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Red parts form the inner crystal.

$${\color{red}\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} \quad \left(\cong \mathbf{B}_2\left(\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}\right)\right)$$

Thm (K.)

Notation as above, we have

$$\mathsf{Sk}_\lambda(x_1, \ldots, x_l, 0, \ldots, 0) = s_\lambda(x).$$

In particular, this is symmetric as an l-variable polynomial.

Robinson correspondence

$$S_n \longrightarrow \bigsqcup_{\lambda \vdash n} ST(\lambda) \times ST(\lambda)$$

$$w \mapsto (P(w), Q(w))$$

is reduced to

skeleton Robinson correspondence

$$\sum_{\lambda \vdash n} \mathrm{Sk}_\lambda(x)\mathrm{Sk}_\lambda(y) = \sum_{w \in S_n} x^{\mathrm{des}P(w)} y^{\mathrm{des}Q(w)}.$$

In particular, $x_i = y_i = 1$ recovers

$$\sum_\lambda f_\lambda^2 = n!.$$

RS correspondence

$$\mathbf{N}^n \longrightarrow \bigsqcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda) \times ST(\lambda)$$

$$w \mapsto (P(w), Q(w))$$

is reduced to **skeleton RS correspondence**:

$$\sum_{\lambda \vdash n} s_\lambda(X)\mathrm{Sk}_\lambda(y) = \sum_{w \in S_n} F_{\mathrm{des}P(w)}(X) y^{\mathrm{des}Q(w)}.$$

Lusztig symmetry

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbf{N}^m$, let

$$\alpha^* = (\alpha_m, \alpha_{m-1}, \ldots, \alpha_1).$$

Thm (K.)

$f_{\lambda\alpha} = f_{\lambda\alpha^*}$. In other words,

$$\mathrm{Sk}_\lambda(x) = \sum_{T \in ST(\lambda)} x^{\mathrm{des}T} = \sum_{T \in ST(\lambda)} x^{(\mathrm{des}T)^*}$$

Idea of a proof:evacuation.

Fact (Young Branching rule)

$$ST(\lambda) = \bigoplus_{\lambda^- \lhd \lambda} ST(\lambda^-).$$

However,

$$\mathrm{Sk}_\lambda(x) = \sum_{\lambda^- \lhd \lambda} \mathrm{Sk}_{\lambda^-}(x)$$

is **not** quite true, but close.

idea

Express this branching with algebra of qsyms, Young lattice and Up/Down operators.

Summary

Part 1

- SSOT
- type-A-to-C-extension of Gessel, Assaf-Searles' theorem

Part 2

- F-, Schur-positivity
- quasi-crystals
- crystal skeleton
- skeleton polynomials

Thanks!