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KL polynomials
Key lemma
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Combinatorial Inequalities of Kazhdan-Lusztig polynomials in Bruhat graphs

Masato Kobayashi

Saitama University

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Key Words

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- Coxeter system $(W, S, T, \ell, <)$

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- Rationally smooth, singular

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- Dyer, Irving, Braden-MacPherson,

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New idea

- Strict edges

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Notation $X(w) = [e, w]$

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Motivation

Understand behavior of

$$P_{uw}(1) \text{ for } u \in X(w)$$

in terms of Bruhat graph.

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Theorem (Kobayashi)

$\exists$ a lower bound of $P_{uw}(1)$ by graph-theoretic distance.

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Idea

When a strict inequality occurs?

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Idea

When a strict inequality occurs?
When

$$P_{uw}(1) > P_{vw}(1) \text{ for } u < v?$$

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$u \longrightarrow w$ means $w = ut$ for some $t \in T$ and $\ell(u) < \ell(w)$.

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Def (Dyer 91)

- Bruhat graph:

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Def (Dyer 91)

- Bruhat graph:
  vertices $w \in W$,
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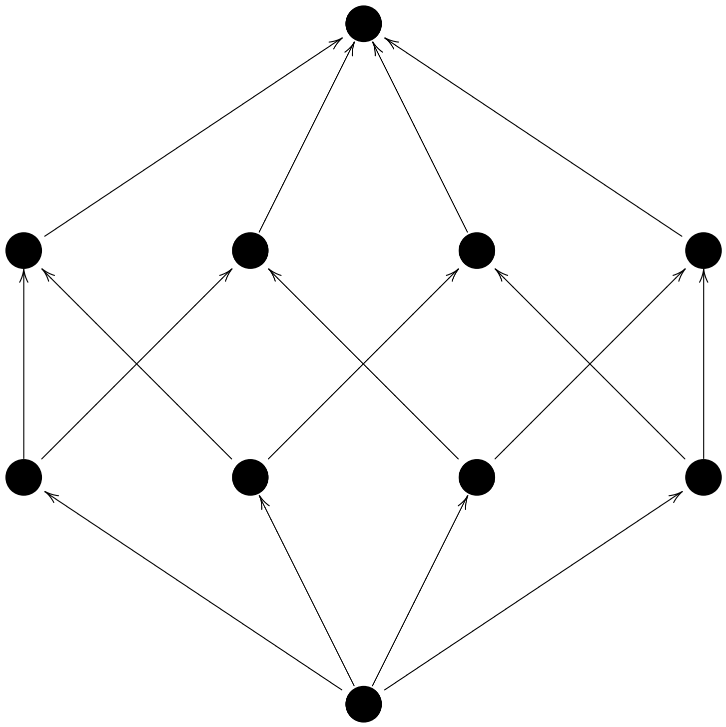
Def (Dyer 91)

- Bruhat graph:
  vertices $w \in W$,
  edges $u \longrightarrow w$.
- Bruhat subgraph for $[u, w]$
- Bruhat path

$$u \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_n = w$$

Figure: [1324, 3412]



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Fact (Kazhdan-Lusztig 79)

$\exists \, \{P_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q]$ (KL polynomials) with

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Fact (Kazhdan-Lusztig 79)

$\exists \{P_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q]$ (KL polynomials) with

1. $P_{uw}(q) = 0$ if $u \not\leq w$,
2. $P_{uw}(q) = 1$ if $u = w$,
3. $\deg P_{uw}(q) \leq (\ell(u,w) - 1)/2$ if $u < w$,
4. if $u \leq w$, then

$$q^{\ell(u,w)} P_{uw}(q^{-1}) = \sum_{u \leq v \leq w} R_{uv}(q) P_{vw}(q),$$

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integer coefficient, but...

W: crystallographic $\Longrightarrow$

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W: crystallographic $\Longrightarrow$

Fact (Irving 88)

All coefficients of KL polynomials in W are nonnegative.

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$\{P_{uw}(1) \mid u \in X(w)\}$: positive integers

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All coefficients of KL polynomials in W are nonnegative.

$\{P_{uw}(1) \mid u \in X(w)\}$: positive integers

Def

$$[u, w] \text{ is } \begin{cases} \text{rationally smooth} & \text{if } P_{uw}(1) = 1, \\ \text{rationally singular} & \text{if } P_{uw}(1) > 1. \end{cases}$$

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W: crystallographic $\Longrightarrow$

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All coefficients of KL polynomials in W are nonnegative.

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$[u, w]$ is $\begin{cases} \text{rationally smooth} & \text{if } P_{uw}(1) = 1, \\ \text{rationally singular} & \text{if } P_{uw}(1) > 1. \end{cases}$

Fact (Braden-MacPherson 01)

$u < v$ in $X(w) \Longrightarrow P_{uw}(q) \geq P_{vw}(q)$ (coefficientwise).

W: crystallographic $\Longrightarrow$

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All coefficients of KL polynomials in W are nonnegative.

$\{P_{uw}(1) \mid u \in X(w)\}$: positive integers

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$$[u, w] \text{ is } \begin{cases} \text{rationally smooth} & \text{if } P_{uw}(1) = 1, \\ \text{rationally singular} & \text{if } P_{uw}(1) > 1. \end{cases}$$

Fact (Braden-MacPherson 01)

$u < v$ in $X(w) \Longrightarrow P_{uw}(q) \geq P_{vw}(q)$ (coefficientwise).

Prop

Let $u < v$ in $X(w)$. Then

$$P_{uw}(q) > P_{vw}(q) \iff P_{uw}(1) > P_{vw}(1).$$

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Idea

Fix $u \in X(w)$. Suppose $P_{uw}(1) > 1$.

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$v = w \implies$ Yes.

Find v "much closer" to u.

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Idea

Fix $u \in X(w)$. Suppose $P_{uw}(1) > 1$.
For which $v \in [u, w]$,

$$P_{uw}(1) > P_{vw}(1) \ ?$$

$v = w \implies$ Yes.

Find v "much closer" to u.

Def (Kobayashi 12)

$u \rightarrow v$ in $[u, w]$ is strict if $P_{uw}(1) > P_{vw}(1)$.

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Key lemma (Kobayashi, to appear)

$P_{uw}(1) > 1$

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Key lemma (Kobayashi, to appear)

$P_{uw}(1) > 1$

$$\implies \exists \text{ strict edge } u \rightarrow v \text{ in } [u, w];$$

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Key lemma (Kobayashi, to appear)

$P_{uw}(1) > 1$

$$\implies \exists \text{ strict edge } u \rightarrow v \text{ in } [u, w];$$

Equivalently:
$\exists t \in T$ *such that*

$$P_{uw}(1) > P_{ut,w}(1) > 0.$$

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(Idea of Proof)

1. first order derivative of R-polynomials

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Key lemma (Kobayashi, to appear)

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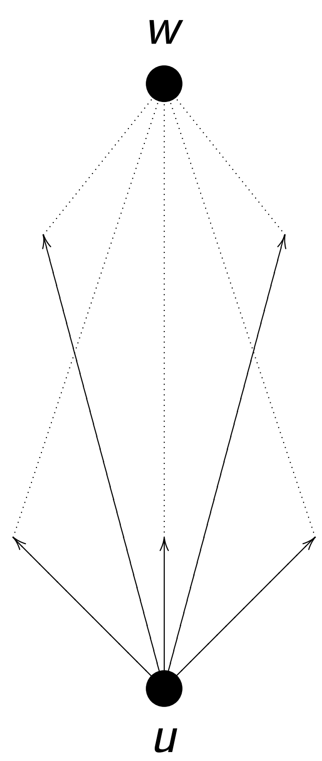
(Idea of Proof)

1. first order derivative of R-polynomials
2. Deodhar's inequality

Figure: existence of a strict edge



$$P_{uw}(1) > P_{ut,w}(1) > 0$$

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$$X_{\text{smooth}}(w) = \{u \in X(w) \mid P_{uw}(1) = 1\}.$$

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$$X_{\mathrm{smooth}}(w) = \{u \in X(w) \mid P_{uw}(1) = 1\}.$$

Def (distance)

$$\mathrm{dist}(u, X_{\mathrm{smooth}}(w))$$
$$= \min\{d \geq 0 \mid u \to v_1 \to \cdots \to v_d \in X_{\mathrm{smooth}}(w)\}.$$

$$X_{\text{smooth}}(w) = \{u \in X(w) \mid P_{uw}(1) = 1\}.$$

Def (distance)

$$\operatorname{dist}(u, X_{\text{smooth}}(w))$$
$$= \min\{d \geq 0 \mid u \to v_1 \to \cdots \to v_d \in X_{\text{smooth}}(w)\}.$$

In particular, $\operatorname{dist}(u, X_{\text{smooth}}(w)) = 0 \iff P_{uw}(1) = 1.$

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Main Thm

Let $u \leq w$ and $d = \mathrm{dist}(u, X_{\mathrm{smooth}}(w))$.

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Main Thm

Let $u \leq w$ and $d = \mathrm{dist}(u, X_{\mathrm{smooth}}(w))$. Then

$$P_{uw}(1) \geq d + 1.$$

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Main Thm

Let $u \leq w$ and $d = \text{dist}(u, X_{\text{smooth}}(w))$. Then

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Suppose $P_{uw}(1) > 1$.

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Let $u \leq w$ and $d = \mathrm{dist}(u, X_{\mathrm{smooth}}(w))$. Then

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Proof

Suppose $P_{uw}(1) > 1$.
Key Lemma $\Longrightarrow \exists$ strict edge $u \rightarrow v_1$ in $X(w)$.

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Let $u \leq w$ and $d = \mathrm{dist}(u, X_{\mathrm{smooth}}(w))$. Then

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Proof

Suppose $P_{uw}(1) > 1$.
Key Lemma $\Longrightarrow$ $\exists$ strict edge $u \longrightarrow v_1$ in $X(w)$.
Repeat: $\exists$ strict edge $v_1 \longrightarrow v_2$ in $X(w)$.

Main Thm

Let $u \leq w$ and $d = \text{dist}(u, X_{\text{smooth}}(w))$. Then

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Proof

Suppose $P_{uw}(1) > 1$.
Key Lemma $\Longrightarrow \exists$ strict edge $u \longrightarrow v_1$ in $X(w)$.
Repeat: $\exists$ strict edge $v_1 \longrightarrow v_2$ in $X(w)$.
Thus

$$u \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_d \quad \text{in } X(w)$$

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Let $u \leq w$ and $d = \mathrm{dist}(u, X_{\mathrm{smooth}}(w))$. Then

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Suppose $P_{uw}(1) > 1$.
Key Lemma $\Longrightarrow \exists$ strict edge $u \longrightarrow v_1$ in $X(w)$.
Repeat: $\exists$ strict edge $v_1 \longrightarrow v_2$ in $X(w)$.
Thus

$$u \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_d \quad \text{in } X(w)$$

$$\underbrace{P_{uw}(1) > P_{v_1 w}(1) > \ \ldots \ > P_{v_d w}(1)}_{d+1}.$$

Main Thm

Let $u \leq w$ and $d = \mathrm{dist}(u, X_{\mathrm{smooth}}(w))$. Then

$$P_{uw}(1) \geq d + 1.$$

Proof

Suppose $P_{uw}(1) > 1$.
Key Lemma $\Longrightarrow \exists$ strict edge $u \rightarrow v_1$ in $X(w)$.
Repeat: $\exists$ strict edge $v_1 \rightarrow v_2$ in $X(w)$.
Thus

$$u \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_d \quad \text{in } X(w)$$

$$\underbrace{P_{uw}(1) > P_{v_1 w}(1) > \ \ldots \ > P_{v_d w}(1)}_{d+1}.$$

Conclude $P_{uw}(1) \geq d + 1$.

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$\mathfrak{g}$: semisimple Lie algebra
Φ: irreducible simply-laced root system
W: Weyl group
Λ^+ : dominant integral weights
$<$: root order

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$\mathfrak{g}$: semisimple Lie algebra
Φ: irreducible simply-laced root system
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Λ^+ : dominant integral weights
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Fact (Stembridge 98)

Let $\lambda, \mu \in \Lambda^+$.

$$\lambda \rhd \mu \Longrightarrow \lambda - \mu = \alpha \, (\exists \, \alpha \ \text{ one positive root})$$

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$\mathfrak{g}$: semisimple Lie algebra
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Any relation to

$$P_{uw}(1) > P_{ut,w}(1) > 0?$$

Conclusion

- Bruhat graphs
- KL polynomials
- rationally singular $\iff$ $P_{uw}(1) > 1$

Conclusion

- Bruhat graphs
- KL polynomials
- rationally singular $\iff$ $P_{uw}(1) > 1$
- Key lemma (= existence of a strict edge)
- Theorem (= a lower bound of $P_{uw}(1)$)

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Reference

M. Kobayashi,

Inequalities on Bruaht graphs, R- and KL polynomials,

(to appear, J. Comb. Th. Ser. A)

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Inequalities on Bruaht graphs, R- and KL polynomials,

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Thank you.