Bruhat graphs

(L polynomials

(ev lemma

Cheorem

Liture work

Combinatorial Inequalities of Kazhdan-Lusztig polynomials in Bruhat graphs

Masato Kobayashi

Saitama University

October 12, 2012

Bruhat graphs

KL polynomials

Kev lemma

Theorem

⁼uture work

Key Words

 $\blacksquare \ \mathsf{Coxeter} \ \mathsf{system} \ (\mathit{W}, \mathit{S}, \mathit{T}, \ell, <)$

- Coxeter system $(W, S, T, \ell, <)$
- Bruhat graphs

- Coxeter system $(W, S, T, \ell, <)$
- Bruhat graphs
- KL polynomials

- Coxeter system $(W, S, T, \ell, <)$
- Bruhat graphs
- KL polynomials
- Rationally smooth, singular

- Coxeter system $(W, S, T, \ell, <)$
- Bruhat graphs
- KL polynomials
- Rationally smooth, singular
- Dyer, Irving, Braden-MacPherson,

- Coxeter system $(W, S, T, \ell, <)$
- Bruhat graphs
- KL polynomials
- Rationally smooth, singular
- Dyer, Irving, Braden-MacPherson,

Key Words

- Coxeter system $(W, S, T, \ell, <)$
- Bruhat graphs
- KL polynomials
- Rationally smooth, singular
- Dyer, Irving, Braden-MacPherson,

New idea

Strict edges

Bruhat graphs

(I polynomials

Cev lemma

Chaaram

uture work

- Introduction
- Bruhat graphs
- 3 KL polynomials
- Key lemma
- 5 Theorem
- **6** Future work

Bruhat graphs

KI polynomials

Kev lemma

Theorem

-- Liture Work Notation X(w) = [e, w]

Bruhat granhs

KL nolynomials

Kay lamma

Thoorom

--uture work Notation X(w) = [e, w]

Motivation

Understand behavior of

$$P_{uw}(1)$$
 for $u \in X(w)$

in terms of Bruhat graph.

Bruhat granhs

KL polynomials

Key lemma

Theorem

⁻uture work

Notation X(w) = [e, w]

Motivation

Understand behavior of

$$P_{uw}(1)$$
 for $u \in X(w)$

in terms of Bruhat graph.

Theorem (Kobayashi)

 \exists a lower bound of $P_{uw}(1)$ by graph-theoretic distance.

Bruhat graphs

XI nolynomials

Key lemma

T1

Futuro work

Notation X(w) = [e, w]

Motivation

Understand behavior of

$$P_{uw}(1)$$
 for $u \in X(w)$

in terms of Bruhat graph.

Theorem (Kobayashi)

 \exists a lower bound of $P_{uw}(1)$ by graph-theoretic distance.

Idea

When a strict inequality occurs?

I heorem

Future work

Notation X(w) = [e, w]

Motivation

Understand behavior of

$$P_{uw}(1)$$
 for $u \in X(w)$

in terms of Bruhat graph.

Theorem (Kobayashi)

 \exists a lower bound of $P_{uw}(1)$ by graph-theoretic distance.

Idea

When a strict inequality occurs?

When

$$P_{uw}(1) > P_{vw}(1)$$
 for $u < v$?

Bruhat graphs

KL polynomials

Kov lomma

_.

Liture work

- Introduction
- Bruhat graphs
- 3 KL polynomials
- Key lemma
- 5 Theorem
- **6** Future work

Bruhat graphs

KL polynomials

Kev lemma

Theorem

-- Liture Work

 $u \to w$ means w = ut for some $t \in T$ and $\ell(u) < \ell(w)$.

Bruhat graphs

KL polynomials

Cev lemma

Theorem

-uture work

 $u \to w$ means w = ut for some $t \in T$ and $\ell(u) < \ell(w)$.

Def (Dyer 91)

■ Bruhat graph:

Bruhat graphs

KL polynomials

Cev lemma

Theorem

-uture work

 $u \to w$ means w = ut for some $t \in T$ and $\ell(u) < \ell(w)$.

Def (Dyer 91)

■ Bruhat graph:

Bruhat graphs

KL polynomials

Kev lemma

Theorem

⁻uture work

 $u \to w$ means w = ut for some $t \in T$ and $\ell(u) < \ell(w)$.

Def (Dyer 91)

■ Bruhat graph: vertices $w \in W$, edges $u \rightarrow w$.

Bruhat graphs

(I polynomials

Kev lemma

-uture work

 $u \to w$ means w = ut for some $t \in T$ and $\ell(u) < \ell(w)$.

Def (Dyer 91)

- Bruhat graph: vertices $w \in W$, edges $u \rightarrow w$.
- Bruhat subgraph for [u, w]

Bruhat graphs

(I polynomials

Kev lemma

_.

-uture work

 $u \to w$ means w = ut for some $t \in T$ and $\ell(u) < \ell(w)$.

Def (Dyer 91)

- Bruhat graph: vertices $w \in W$, edges $u \rightarrow w$.
- Bruhat subgraph for [u, w]
- Bruhat path

$$u \rightarrow v_1 \rightarrow \cdots \rightarrow v_n = w$$

Bruhat graphs

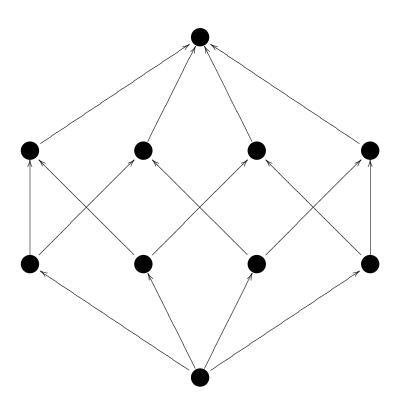
KL polynomials

Key lemma

Theorem

-- uture work

Figure: [1324, 3412]



Bruhat graphs

KL polynomials

Kev lemma

Liture work

- Introduction
- Bruhat graphs
- **3** KL polynomials
- Key lemma
- 5 Theorem
- **6** Future work

Bruhat graphs
KL polynomials

Fact (Kazhdan-Lusztig 79)

 $\exists \ \{P_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q] \ (\mathsf{KL} \ \mathsf{polynomials}) \ \mathsf{with}$

Bruhat graphs

KL polynomials

Key Jemma

toy forming

Fact (Kazhdan-Lusztig 79)

- $\exists \{P_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q] \text{ (KL polynomials) with}$
 - $P_{uw}(q) = 0 \text{ if } u \not\leq w,$
 - $P_{uw}(q) = 1 \text{ if } u = w,$
 - **3** $\deg P_{uw}(q) \le (\ell(u, w) 1)/2 \text{ if } u < w$,
 - $\underline{\mathbf{u}}$ if $u \leq w$, then

$$q^{\ell(u,w)}P_{uw}(q^{-1}) = \sum_{u \leq v \leq w} R_{uv}(q)P_{vw}(q),$$

3ruhat graphs

KL polynomials

, ,

,

Fact (Kazhdan-Lusztig 79)

 $\exists \{P_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q] \text{ (KL polynomials) with}$

$$P_{uw}(q) = 0 \text{ if } u \not\leq w,$$

$$P_{uw}(q) = 1 \text{ if } u = w,$$

3
$$\deg P_{uw}(q) \le (\ell(u, w) - 1)/2 \text{ if } u < w$$
,

 $\underline{\mathbf{u}}$ if $u \leq w$, then

$$q^{\ell(u,w)}P_{uw}(q^{-1}) = \sum_{u \leq v \leq w} R_{uv}(q)P_{vw}(q),$$

integer coefficient, but...

Bruhat grap

KL polynomials

Key lemma

T1

Euturo work

W: crystallographic \Longrightarrow

Introduction
Bruhat graphs
KL polynomials

Key lemma

Theorem

-- Liture work W: crystallographic \Longrightarrow

Fact (Irving 88)

All coefficients of KL polynomials in $\it W$ are nonnegative.

KL polynomials

W: crystallographic \Longrightarrow

Fact (Irving 88)

All coefficients of KL polynomials in W are nonnegative.

 $\{P_{uw}(1) \mid u \in X(w)\}$: positive integers

KL polynomials

W: crystallographic \Longrightarrow

Fact (Irving 88)

All coefficients of KL polynomials in W are nonnegative.

$$\{P_{uw}(1) \mid u \in X(w)\}$$
: positive integers

Def

$$[u,w]$$
 is $egin{cases} ext{rationally smooth} & ext{if } P_{uw}(1)=1, \ ext{rationally singular} & ext{if } P_{uw}(1)>1. \end{cases}$

Introduction
Bruhat graphs
KL polynomials
Key lemma

W: crystallographic \Longrightarrow

Fact (Irving 88)

All coefficients of KL polynomials in W are nonnegative.

$$\{P_{uw}(1) \mid u \in X(w)\}$$
: positive integers

Def

$$egin{aligned} [u,w] ext{ is } & egin{aligned} ext{rationally smooth} & ext{if } P_{uw}(1)=1, \ ext{rationally singular} & ext{if } P_{uw}(1)>1. \end{aligned}$$

Fact (Braden-MacPherson 01)

$$u < v \text{ in } X(w) \Longrightarrow P_{uw}(q) \ge P_{vw}(q) \text{ (coefficientwise)}.$$

Introduction
Bruhat graphs
KL polynomials
Key lemma

W: crystallographic \Longrightarrow

Fact (Irving 88)

All coefficients of KL polynomials in W are nonnegative.

$$\{P_{uw}(1) \mid u \in X(w)\}$$
: positive integers

Def

$$egin{aligned} [u,w] ext{ is } & egin{aligned} ext{rationally smooth} & ext{if } P_{uw}(1)=1, \ ext{rationally singular} & ext{if } P_{uw}(1)>1. \end{aligned}$$

Fact (Braden-MacPherson 01)

$$u < v \text{ in } X(w) \Longrightarrow P_{uw}(q) \ge P_{vw}(q) \text{ (coefficientwise)}.$$

Prop

Let
$$u < v$$
 in $X(w)$. Then

$$P_{uw}(q) > P_{vw}(q) \iff P_{uw}(1) > P_{vw}(1).$$

Bruhat graphs

KL polynomials

Key lemma

[hoorom

uture work

- Introduction
- Bruhat graphs
- **3** KL polynomials
- Key lemma
- 5 Theorem
- **6** Future work

Bruhat graphs

KL polynomials

Key lemma

Theorem

--uture work

ldea

Fix $u \in X(w)$. Suppose $P_{uw}(1) > 1$.

Bruhat graphs

(I polynomials

Key lemma

Theorem

⁻uture work

ldea

Fix $u \in X(w)$. Suppose $P_{uw}(1) > 1$. For which $v \in [u, w]$,

$$P_{uw}(1) > P_{vw}(1)$$
 ?

Bruhat graphs

Al nolynomials

Key lemma

Theorem

⁻uture work

ldea

Fix $u \in X(w)$. Suppose $P_{uw}(1) > 1$. For which $v \in [u, w]$,

$$P_{uw}(1) > P_{vw}(1)$$
 ?

 $v = w \Longrightarrow \text{Yes.}$

Bruhat graphs

KL polynomials

Key lemma

Theorem

ldea

Fix $u \in X(w)$. Suppose $P_{uw}(1) > 1$. For which $v \in [u, w]$,

$$P_{uw}(1) > P_{vw}(1)$$
 ?

 $v = w \Longrightarrow \text{Yes.}$

Find v "much closer" to u.

Bruhat granhs

Al nolynomials

Key lemma

Theorem

⁻uture work

Idea

Fix $u \in X(w)$. Suppose $P_{uw}(1) > 1$. For which $v \in [u, w]$,

$$P_{uw}(1) > P_{vw}(1)$$
 ?

 $v = w \Longrightarrow \text{Yes.}$

Find v "much closer" to u.

Def (Kobayashi 12)

 $u \rightarrow v$ in [u, w] is strict if $P_{uw}(1) > P_{vw}(1)$.

Bruhat graphs

KL polynomials

Key lemma

Theorem

Eutura warl

Key lemma (Kobayashi, to appear)

$$P_{uw}(1) > 1$$

Bruhat graphs

KL polynomials

Key lemma

Theorem

Eutura warl

Key lemma (Kobayashi, to appear)

$$P_{uw}(1) > 1$$

 $\implies \exists \text{ strict edge } u \rightarrow v \text{ in } [u, w];$

Bruhat graphs

XI nolynomials

Key lemma

Theorem

Future work

Key lemma (Kobayashi, to appear)

$$P_{uw}(1) > 1$$

$$\implies \exists$$
 strict edge $u \rightarrow v$ in $[u, w]$;

Equivalently:

 $\exists t \in T$ such that

$$P_{uw}(1) > P_{ut,w}(1) > 0.$$

Bruhat graphs

(L polynomials

Key lemma

Theorem

Eutura wark

Key lemma (Kobayashi, to appear)

$$P_{uw}(1) > 1$$

$$\implies \exists$$
 strict edge $u \rightarrow v$ in $[u, w]$;

Equivalently:

 $\exists t \in T \text{ such that }$

$$P_{uw}(1) > P_{ut,w}(1) > 0.$$

(Idea of Proof)

 \blacksquare first order derivative of R-polynomials

Bruhat granhs

KL polynomials

Key lemma

Theorem

Eutura wark

Key lemma (Kobayashi, to appear)

$$P_{uw}(1) > 1$$

$$\implies \exists$$
 strict edge $u \rightarrow v$ in $[u, w]$;

Equivalently:

 $\exists t \in T \text{ such that }$

$$P_{uw}(1) > P_{ut,w}(1) > 0.$$

(Idea of Proof)

- \blacksquare first order derivative of R-polynomials
- Deodhar's inequality

Bruhat graphs

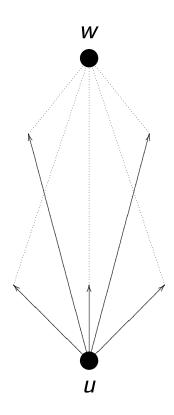
Al nolynomials

Key lemma

Cheorem

-Liture work

Figure: existence of a strict edge



$$P_{uw}(1) > P_{ut,w}(1) > 0$$

Bruhat graphs

KL nolynomials

Cev lemma

Theorem

Liture work

- Introduction
- Bruhat graphs
- KL polynomials
- Key lemma
- 5 Theorem
- **6** Future work

Bruhat graphs

KL polynomials

Kev lemma

Theorem

-- Liture Work

$$X_{\sf smooth}(w) = \{u \in X(w) \mid P_{uw}(1) = 1\}.$$

Bruhat graphs

(L nolynomials

(ev lemma

Theorem

⁻uture work

$$X_{\text{smooth}}(w) = \{ u \in X(w) \mid P_{uw}(1) = 1 \}.$$

Def (distance)

$$\operatorname{dist}(u, X_{\operatorname{smooth}}(w))$$

$$= \min\{d \geq 0 \mid u \rightarrow v_1 \rightarrow \cdots \rightarrow v_d \in X_{\operatorname{smooth}}(w)\}.$$

Bruhat graphs

KL polynomials

Cev lemma

Theorem

-uture work

$$X_{\text{smooth}}(w) = \{ u \in X(w) \mid P_{uw}(1) = 1 \}.$$

Def (distance)

$$\operatorname{dist}(u, X_{\operatorname{smooth}}(w))$$

$$= \min\{d \geq 0 \mid u \rightarrow v_1 \rightarrow \cdots \rightarrow v_d \in X_{\operatorname{smooth}}(w)\}.$$

In particular, $dist(u, X_{smooth}(w)) = 0 \iff P_{uw}(1) = 1$.

Introduction

Rrubat graph

KL polynomials

Kev lemma

Theorem

Future work

Main Thm

Let $u \leq w$ and $d = dist(u, X_{smooth}(w))$.

Bruhat graphs

KL polynomials

Kev lemma

Theorem

-- Inture Work

Main Thm

Let $u \leq w$ and $d = dist(u, X_{smooth}(w))$. Then

$$P_{uw}(1) \geq d+1$$
.

Bruhat graphs

KL polynomials

Kev lemma

Theorem

⁼uture worl

Main Thm

Let $u \leq w$ and $d = dist(u, X_{smooth}(w))$. Then

$$P_{uw}(1) \ge d + 1.$$

Proof

Suppose $P_{uw}(1) > 1$.

Bruhat graphs

KL polynomials

ev lemma

Theorem

⁻uture work

Main Thm

Let $u \leq w$ and $d = dist(u, X_{smooth}(w))$. Then

$$P_{uw}(1) \ge d + 1$$
.

Proof

Suppose $P_{uw}(1) > 1$.

Key Lemma $\Longrightarrow \exists$ strict edge $u \rightarrow v_1$ in X(w).

Bruhat graphs

KL polynomials

Key Jemma

Theorem

⁼uture worl

Main Thm

Let $u \leq w$ and $d = dist(u, X_{smooth}(w))$. Then

$$P_{uw}(1) \ge d + 1.$$

Proof

Suppose $P_{uw}(1) > 1$.

Key Lemma $\Longrightarrow \exists$ strict edge $u \rightarrow v_1$ in X(w).

Repeat: \exists strict edge $v_1 \rightarrow v_2$ in X(w).

Bruhat granhs

XI nolynomials

Cev lemma

Theorem

⁼uture work

Main Thm

Let $u \leq w$ and $d = dist(u, X_{smooth}(w))$. Then

$$P_{uw}(1) \ge d + 1.$$

Proof

Suppose $P_{uw}(1) > 1$.

Key Lemma $\Longrightarrow \exists$ strict edge $u \rightarrow v_1$ in X(w).

Repeat: \exists strict edge $v_1 \rightarrow v_2$ in X(w).

Thus

$$u \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_d$$
 in $X(w)$

Bruhat granhs

(L nolynomials

(ev lemma

Theorem

⁼uture work

Main Thm

Let $u \leq w$ and $d = dist(u, X_{smooth}(w))$. Then

$$P_{uw}(1) \ge d + 1.$$

Proof

Suppose $P_{uw}(1) > 1$.

Key Lemma $\Longrightarrow \exists$ strict edge $u \rightarrow v_1$ in X(w).

Repeat: \exists strict edge $v_1 \rightarrow v_2$ in X(w).

Thus

$$u \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_d$$
 in $X(w)$

$$\underbrace{P_{uw}(1) > P_{v_1w}(1) > \ldots > P_{v_dw}(1)}_{d+1}.$$

Bruhat granhs

(L polynomials

Kay lamma

Theorem

Main Thm

Let $u \leq w$ and $d = dist(u, X_{smooth}(w))$. Then

$$P_{uw}(1) \ge d + 1.$$

Proof

Suppose $P_{uw}(1) > 1$.

Key Lemma $\Longrightarrow \exists$ strict edge $u \rightarrow v_1$ in X(w).

Repeat: \exists strict edge $v_1 \rightarrow v_2$ in X(w).

Thus

$$u \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_d$$
 in $X(w)$

$$\underbrace{P_{uw}(1) > P_{v_1w}(1) > \ldots > P_{v_dw}(1)}_{d+1}.$$

Conclude $P_{uw}(1) \ge d + 1$.

Bruhat graphs

KL polynomials

Kay lamma

Chaaram

Future work

- Introduction
- Bruhat graphs
- **3** KL polynomials
- Key lemma
- 5 Theorem
- **6** Future work

Bruhat graphs

Al nolynomials

Kev lemma

Theorem

Future work

g: semisimple Lie algebra

Φ: irreducible simply-laced root system

W: Weyl group

 Λ^+ : dominant integral weights

<: root order

Bruhat graphs

KL polynomials

Key lemma

_.

Future work

g: semisimple Lie algebra

Φ: irreducible simply-laced root system

W: Weyl group

 Λ^+ : dominant integral weights

<: root order

Fact (Stembridge 98)

Let $\lambda, \mu \in \Lambda^+$.

 $\lambda \rhd \mu \Longrightarrow \lambda - \mu = \alpha \, (\exists \, \alpha \, \text{ one positive root})$

Bruhat granhs

(L polynomials

_.

Future work

g: semisimple Lie algebra

Φ: irreducible simply-laced root system

W: Weyl group

 Λ^+ : dominant integral weights

<: root order

Fact (Stembridge 98)

Let $\lambda, \mu \in \Lambda^+$.

$$\lambda \rhd \mu \Longrightarrow \lambda - \mu = \alpha \, (\exists \, \alpha \, \text{ one positive root})$$

Any relation to

$$P_{uw}(1) > P_{ut,w}(1) > 0$$
?

Bruhat graphs

XI nolynomials

Kev lemma

_.

Future work

Conclusion

- Bruhat graphs
- KL polynomials
- lacktriangleright rationally singular $\iff P_{\mathit{uw}}(1) > 1$

Bruhat graphs

(I polynomials

Cev lemma

The engine

Future work

Conclusion

- Bruhat graphs
- KL polynomials
- lacktriangleright rationally singular $\iff P_{\mathit{uw}}(1) > 1$
- Key lemma (= existence of a strict edge)
- Theorem (= a lower bound of $P_{uw}(1)$)

Bruhat graphs

KL nolynomials

Key lemma

Cheorem

Future work

Reference

M. Kobayashi,

Inequalities on Bruaht graphs, R- and KL polynomials,

(to appear, J. Comb. Th. Ser. A)

Bruhat graphs

KL polynomials

Kev lemma

Chaaram

Future work

Reference

M. Kobayashi,

Inequalities on Bruaht graphs, R- and KL polynomials,

(to appear, J. Comb. Th. Ser. A)

Thank you.