q-determinant, q-Vandermonde and signed bigrassmannian polynomials

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March 2021

1. **Introduction**
2. q-determinant
3. q-Vandermonde
4. Proof of Main Theorem
5. Why signed bigrassmannian permutation polynomials?

This work is a byproduct of my research on alternating sign matrices at a crossroad of

- enumerative combinatorics,
- group theory,
- order theory,
- linear algebra.

Definition

For $w \in S_n$ ($n \geq 2$), say (i, j) is an **inversion** of w if

$$i < j \text{ and } w(i) > w(j).$$

The **length** of w is

$$\ell(w) = |\{(i, j) \mid i < j \text{ and } w(i) > w(j)\}|.$$

The **sign** of w is $(-1)^{\ell(w)}$.

Fact (Linear Algebra)

$$\sum_{w \in S_n} (-1)^{\ell(w)} = 0.$$

What is a q-analog of this? I found this:

$$\sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)} = \prod_{1 \le i < j \le n} (1 - q^{j-i})$$

where

$$\beta(w) = \sum_{i,j:\text{inversion of } w} (j - i)$$

is the **bigrassmannian statistics**.

Example

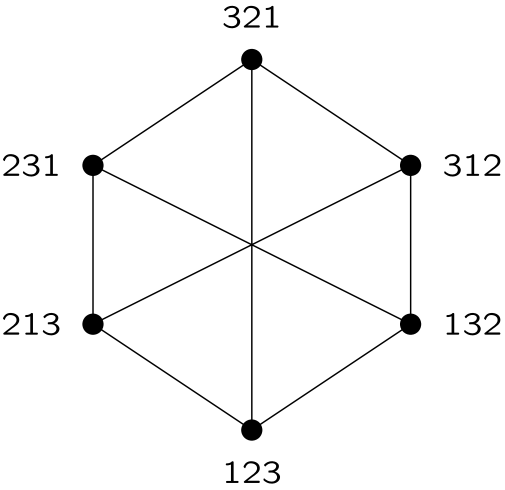
$$w = 312.$$

We have

$$\ell(w) = 2$$

with inversions (3, 1), (3, 2) and moreover

$$\beta(w) = (3-1) + (3-2) = 3.$$



$\ell = 3$, $\beta = 4$

$\ell = 2$, $\beta = 3$

$\ell = 1$, $\beta = 1$

$\ell = 0$, $\beta = 0$

Definition

$$B_n(q) = \sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)}.$$

Call this **signed bigrassmannian polynomial**.

Main theorem (Kobayashi 2015)

For $n \geq 2$,

$$B_n(q) = \prod_{1 \leq i < j \leq n} (1 - q^{j-i})$$

How to prove this? Need “q-det".

Fact (Kobayashi 2011)

$$\beta(w) = \frac{1}{2}\sum_{i=1}^{n}(w(i) - i)^2.$$

Here, $w(i), i$ appear together!

q-determinant

For a square matrix $A = (a_{ij})$, let

$$\det_q(a_{ij}) = \det(q^{(j-i)^2/2}a_{ij}).$$

For example,

$$\det_q \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & q^{1/2} & q^{4/2} \\ q^{1/2} & 1 & q^{1/2} \\ q^{4/2} & q^{1/2} & 1 \end{pmatrix}.$$

We can rephrase

$$\sum_{w \in S_n} (-1)^{\ell(w)} = 0$$

as

$$\det(1)_{i,j=1}^{n} = 0.$$

A q-analog of this is

$$\det_q(1)_{i,j=1}^{n} = \sum_{w \in S_n} (-1)^{\ell(w)} q^{(w(1)-1)^2/2} \cdots q^{(w(n)-n)^2/2}$$

$$= \sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)} = B_n(q).$$

How to compute $\det_q(1)_{i,j=1}^{n}$? $\rightarrow$ Vandermonde

Classical Vandermonde

$$\det(x_i^{j-1})_{i,j=1}^n = \prod_{1\le i<j\le n} (x_j - x_i).$$

Its q-analog (Kobayashi)

$$\det_q(x_i^{j-1})_{i,j=1}^n = \prod_{1\le i<j\le n} (x_j - q^{j-i} x_i).$$

For example,

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

while

$$\det_q \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = (x_2 - qx_1)(x_3 - q^2 x_1)(x_3 - qx_2).$$

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Lemma (Bressoud 1999)

$$\prod_{1\le i<j\le n}(1-q^{j-i}) = \sum_{w\in S_n}(-1)^{\ell(w)} q^{\sum_{i,j:\text{inversion of } w}(j-i)}$$

Proof of Main theorem

Let $x_i = 1$ for all i in q-Vandermonde. Then

$\det_q(1) = \prod_{1\le i<j\le n}(1-q^{j-i}) = \sum_{w\in S_n}(-1)^{\ell(w)} q^{\sum_{i,j:\text{inversion of } w}(j-i)}$.

This is nothing but $B_n(q)$, as required.

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Why signed bigrassmannian polynomials?

Because both $\ell(w), \beta(w)$ play a crucial role for the poset structure of S_n as follows.

Definition

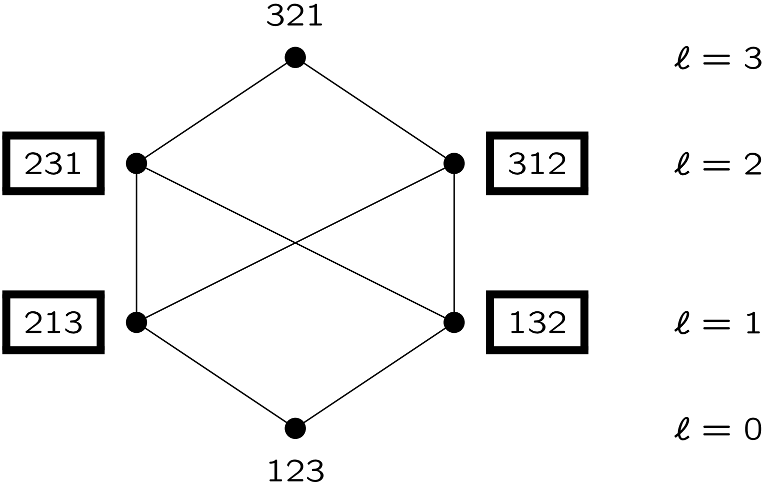
Say $w \in S_n$ is **bigrassmannian** if there exists a unique pair $(i, j) \in \{1, 2, \ldots, n-1\}^2$ such that $w^{-1}(i) > w^{-1}(i+1)$ and $w(j) > w(j+1)$.

Definition

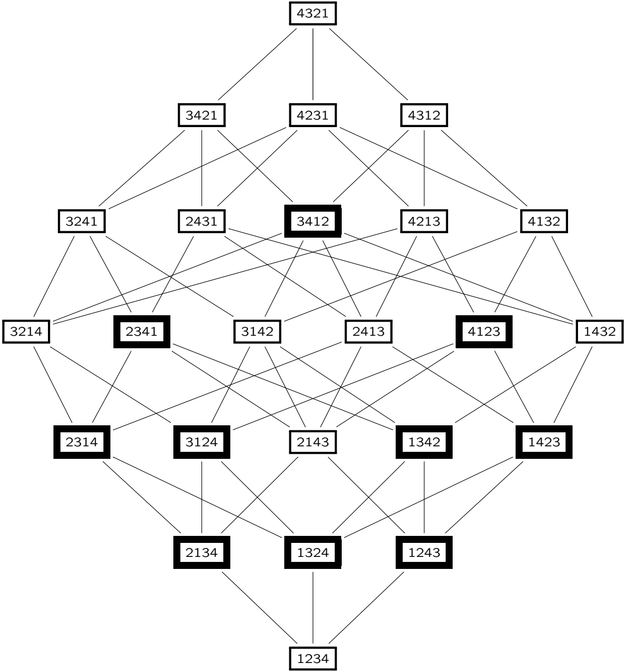
Define **Bruhat order** $\leq$ on S_n as the transitive closure of the following binary relation: $v \to w$ meaning $w = vt_{ij}$, for some $i < j$, t_{ij} a transposition and $\ell(v) < \ell(w)$.

Fact

$(S_n, \leq, \ell)$ is a graded poset.



with $\boxed{}$ bigrassmannian permutations.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Bigrassmannian permutations play an important role in Bruhat order.

Definition

Let P be a poset and $w \in P$. Say w is **join-irreducible** if

1. w is not the minimum of P.
2. $w = u_1 \vee \cdots \vee u_k \Longrightarrow w = u_i$ for some i.

Fact (Lascoux-Schützenberger 1996)

For $w \in S_n$, the following are equivalent:

1. w is bigrassmannian.
2. w is join-irreducible.

A finite lattice $(L, \leq, \vee, \wedge)$ is **distributive** if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in L$.

Fact

In a finite distributive lattice L, each $w \in L$ ($w \neq \min L$) can be uniquely written as

$$w = u_1 \vee \cdots \vee u_k$$

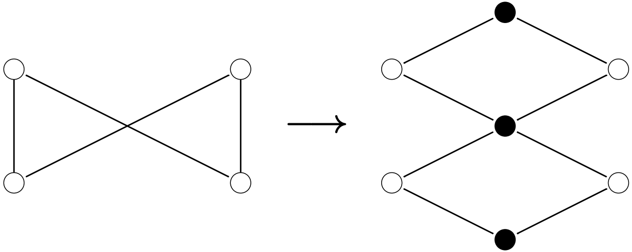
where u_i is join-irreducible.

Fact (MacNeille 1937)

If P is a finite poset, then there exists the smallest distributive lattice $L(P)$ containing P.

Call $L(P)$ the **MacNeille completion** of P.

Figure: An example of MacNeille completion



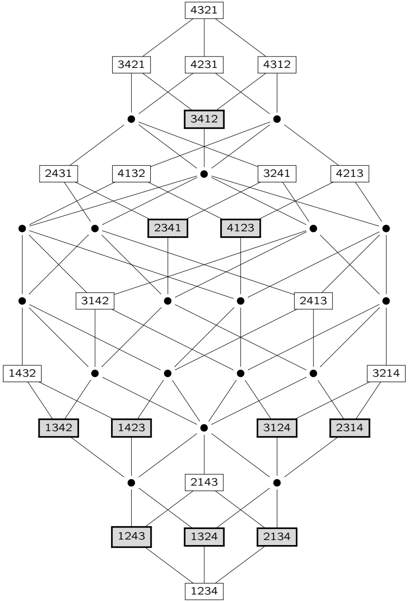
Fact

Every finite distributive lattice L is a graded poset ranked by

$$\beta(w) = |\{u \in L \mid u \leq w, u \text{ is join-irreducible}\}|.$$

Consequence

There exists an extension of Bruhat order $\leq$ such that $(L(S_n), \leq, \beta)$ is graded.



$$B_4(q) = (1-q)^3(1-q^2)^2(1-q^3) = 1 - 3q + q^2 + 4q^3 - 2q^4 - 2q^5 - 2q^6 + 4q^7 + q^8 - 3q^9 + q^{10}$$

Summary

$$\underbrace{(S_n, \leq, \ell)}_{\textit{graded}} \xrightarrow{\textit{MacNeille completion}} \underbrace{(L(S_n), \leq, \beta)}_{\textit{graded}}$$

$$\underbrace{\sum_{w \in S_n} (-1)^{\ell(w)} = 0}_{\det(1)} \xrightarrow{q\text{-analog}} \underbrace{\color{red}{\sum_{\mathbf{w} \in \mathbf{S_n}} (-\mathbf{1})^{\ell(\mathbf{w})} \mathbf{q}^{\beta(\mathbf{w})} = \prod_{\mathbf{1} \leq \mathbf{i} < \mathbf{j} \leq \mathbf{n}} (\mathbf{1} - \mathbf{q}^{\mathbf{j}-\mathbf{i}})}}_{\det_q(1)}$$

Next?

1. S_n is a Coxeter group of type A. Do the same for type BC, D, E, ...
2. Find a combinatorial interpretation of q-Vandermonde.

Thanks!

- M. Kobayashi, Enumeration of bigrassmannian permutations below a permutation in Bruhat order, Order 28 (2011), no. 1, 131-137.
- M. Kobayashi, Enumerative Combinatorics of determinants and signed bigrassmannian polynomials, Okayama Math. J. 57 (2015), 159-172.