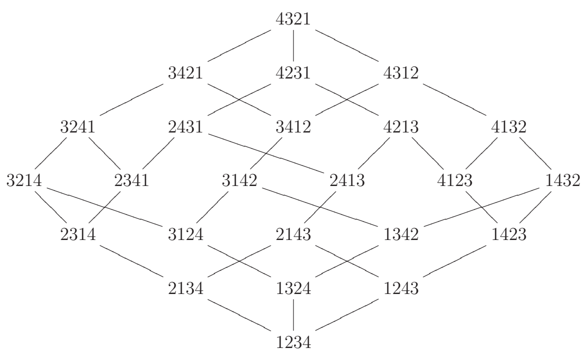
COMBINATORICS ON BIGRASSMANNIAN PERMUTATIONS AND ESSENTIAL SETS

MASATO KOBAYASHI

Contents



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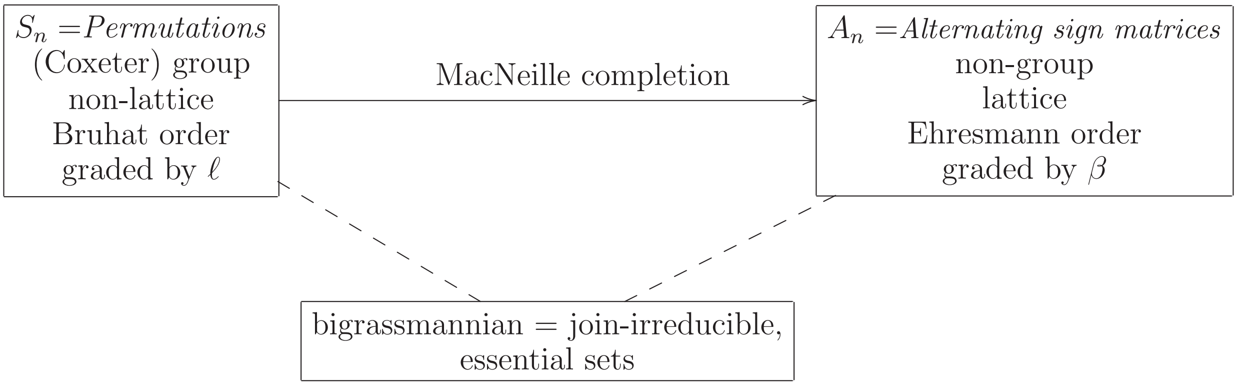


FIGURE 1. Background

1. SYMMETRIC GROUPS

Introduction.

S_n as a Coxeter group.

Notation.

$W =$ $\qquad\qquad$ $S =$

$T =$ $\qquad\qquad$ $\ell =$

Inversion set

$I(w) =$

Def (height, weak order, Bruhat order)**.**

Permutation statistics:

Coxeter length major index

descent exc

MacMahon:

$$\sum_{w \in S_n} q^{\ell(w)} =$$

Mahonian, Eulerian, $\boxed{}$.

Bigrassmannian permutations? [Four-box construction] Rule

▷

▷

$\boxed{123\cdots n} \to \boxed{}\boxed{}\boxed{}\boxed{}$.

$\boxed{1}\boxed{34567}\boxed{2}\boxed{8}$, $\boxed{}\boxed{}\boxed{}\boxed{}$, $\boxed{}\boxed{}\boxed{}\boxed{}$ and $\boxed{}\boxed{}\boxed{}\boxed{}$.

Def (bigrassmanninan permutations)**.**

Question.

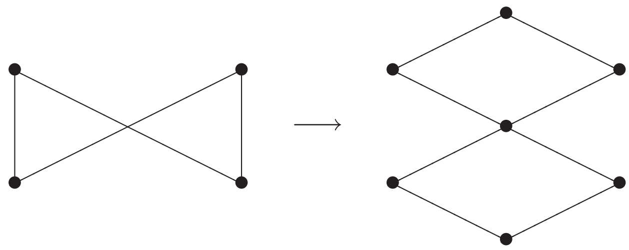
Def (join-irreducibility)**.**

Fact (Lascoux-Schützenberger 1996)**.** The following are equivalent:

(1)

(2)

FIGURE 2. MacNeille completion



Bigrassmannian statistics.

Def (bigrassmannian statistic)**.**

$$\beta(w) =$$

Fact (Kobayashi 2011)**.**

$$\beta(w) = \sum_{\substack{i<j \\ w(i)>w(j)}} (w(i) - w(j)) = \frac{1}{2}\sum_{i=1}^{n}(i - w(i))^2.$$

Example.

$$\beta(3412) =$$

$$\beta(4321) =$$

Def (signed bigrassmannian statistic)**.**

$$B_n(q) =$$

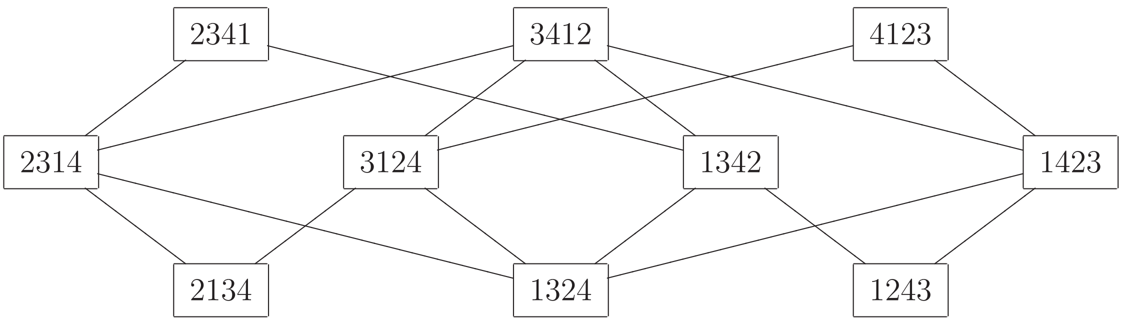


FIGURE 3. Bruhat order of bigrassmannian permutations

Signed bigrassmannian statistics.

Def (q-analog of a matrix).

Proposition (determinantal expression).

Example.

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}_q = \det \begin{pmatrix} 1 & q^{1/2} & q^{4/2} & q^{9/2} \\ q^{1/2} & 1 & q^{1/2} & q^{4/2} \\ q^{4/2} & q^{1/2} & 1 & q^{1/2} \\ q^{9/2} & q^{4/2} & q^{1/2} & 1 \end{pmatrix}$$

$$=$$

Theorem. $B_n(q) =$

(Open problem)

-
-

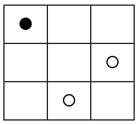
Table 1. signed bigrassmannian statistic over S_4

	sign	β		sign	β		sign	β		sign	β
1234	$+$	0	2134	$-$	1	3124	$+$	3	4123	$-$	6
1243	$-$	1	2143	$+$	2	3142	$-$	5	4132	$+$	7
1324	$-$	1	2314	$+$	3	3214	$-$	4	4213	$+$	7
1342	$+$	3	2341	$-$	6	3241	$+$	7	4231	$-$	9
1423	$+$	3	2413	$-$	5	3412	$+$	8	4312	$-$	9
1432	$-$	4	2431	$+$	7	3421	$-$	9	4321	$+$	10

2. Essential sets

Essential sets and Baxter permutations.

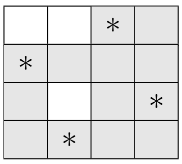
Def (colored diagram)**.**



Def (Rothe diagram, essential set)**.**

$D(w) = ume$

$\mathrm{Ess}(w) = ume$



Def (Baxter permutations)**.**

Fact (Eriksson-Linusson)**.** For $w \in S_n$, the following are equivalent:

(1) w is Baxter.

(2)

Corollary. *bigrassmannian* $\Longrightarrow$ *Baxter.*

dual essential sets.

Def (dual essential set)**.**

Def (essential diagram)**.**

Theorem (Kobayashi 2013)**.** *The following are equivalent:*

(1) x *is Baxter.*

(2)

Theorem (cluster-like structure, Kobayashi 2013)**.** *If* $y = xs_i$, $y(i) > y(i+1)$ *and* x, y *Baxter, then*

local move

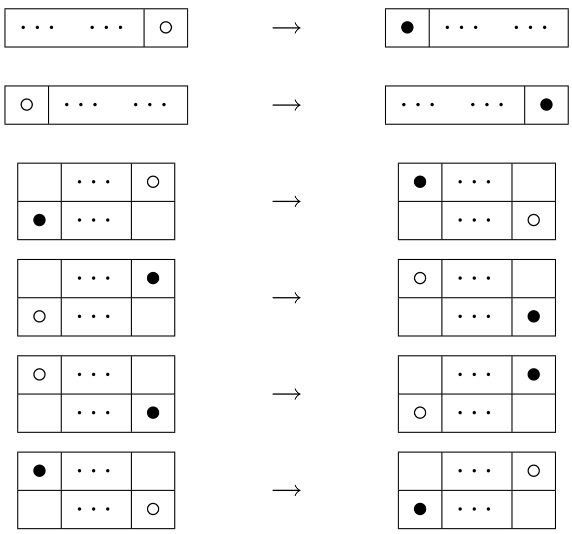
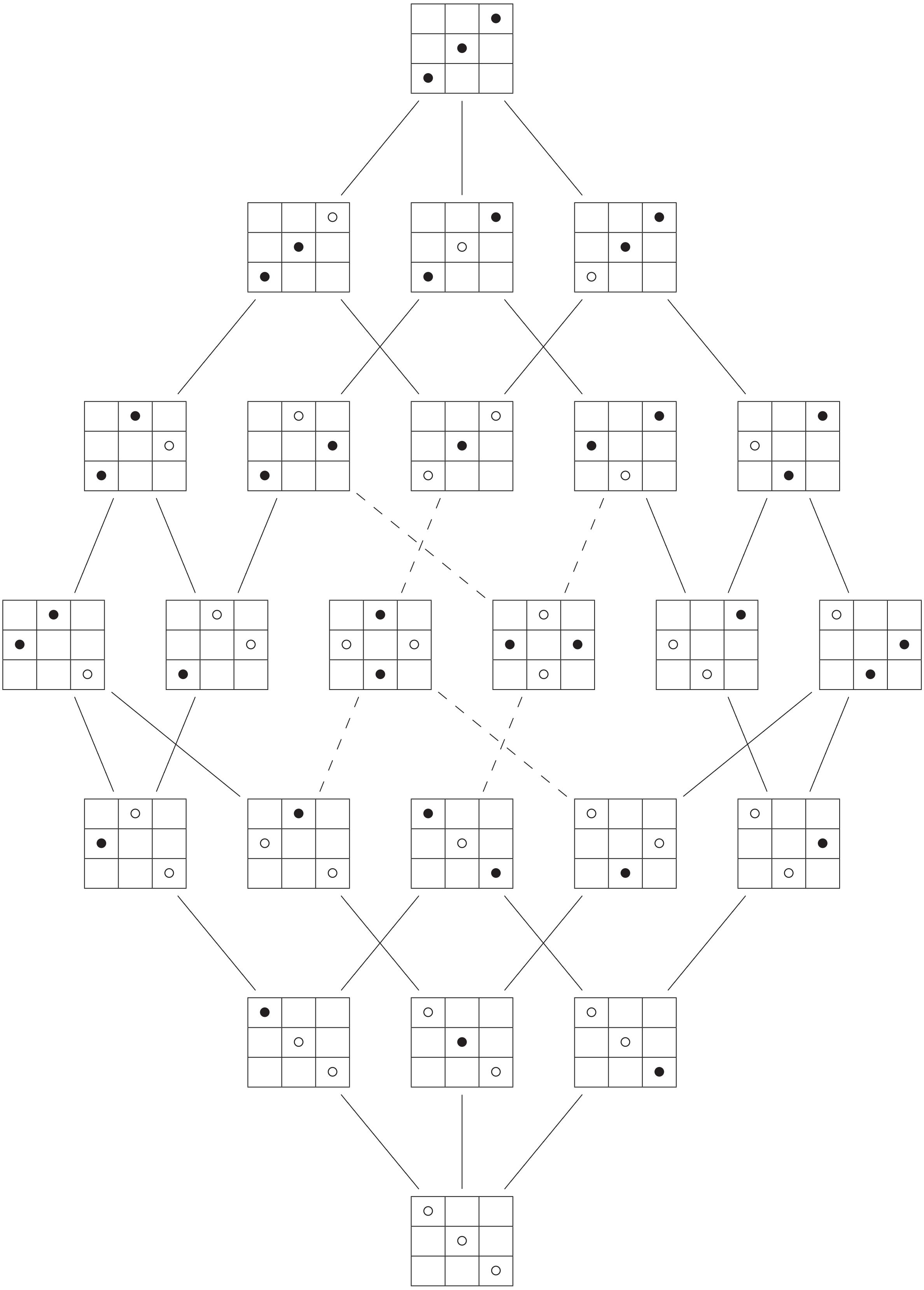


FIGURE 4. Essential diagrams on S_4



Ranked diagrams.

Def (ranked diagram)**.**

$\widetilde{x}(i,j) =$

$\widetilde{x} = (\widetilde{x}(i,j))_{i,j=1}^{n-1}$

$$231 = \begin{array}{|c|c|c|}\hline & * & \\\hline & & * \\\hline * & & \\\hline\end{array} \to \widetilde{231} = \begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\\hline 0 & 1 & 2 \\\hline 1 & 2 & 3 \\\hline\end{array}.$$

Proposition (Essential conditions)**.** *Let $x \in S_n$ and $(i,j) \in [n-1]^2$. Then*

$$\begin{array}{lll} (1) & j < x(i) & \iff \widetilde{x}(i-1,j) = \widetilde{x}(i,j). \\ (2) & i < x^{-1}(j) & \iff \widetilde{x}(i,j-1) = \widetilde{x}(i,j). \\ (3) & x(i+1) \leq j & \iff \widetilde{x}(i+1,j) = \widetilde{x}(i,j)+1. \\ (4) & x^{-1}(j+1) \leq i & \iff \widetilde{x}(i,j+1) = \widetilde{x}(i,j)+1. \end{array}$$

Proposition. *The following are equivalent:*

(1) $x \leq y$
(2) $\widetilde{x}(i,j) \geq \widetilde{y}(i,j)$ *for all* i,j.

Theorem (Kobayashi)**.** *The following are equivalent:*

(1) $x \leq y$
(2) $\widetilde{x}(i,j) \geq \widetilde{y}(i,j)$ *for all* $(i,j) \in \mathrm{Ess}(x)$.

Example. $\widetilde{13254} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\\hline 1 & 1 & 2 & 2 \\\hline 1 & 2 & 3 & 3 \\\hline 1 & 2 & 3 & 3 \\\hline\end{array}$ and $\widetilde{35241} = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\\hline 0 & 0 & 1 & 1 \\\hline 0 & 1 & 2 & 2 \\\hline 0 & 1 & 2 & 3 \\\hline\end{array}$.

$\mathrm{Ess}(13254) = \{\}$

$$F(13254) = \begin{array}{|c|c|c|c|}\hline & & & \\\hline & 1 & & \\\hline & & & \\\hline & & & 3 \\\hline\end{array} \text{ and } \widetilde{35241}|_{\mathrm{Ess}(13254)} = \begin{array}{|c|c|c|c|}\hline & & & \\\hline & 0 & & \\\hline & & & \\\hline & & & 3 \\\hline\end{array}$$

Theorem. *Kobayashi*

$\mathrm{Max}\, B_n(w) \cong \mathrm{Ess}(w)$

3. Alternating sign matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

alternating sign matrices monotone triangles

Def (Alternating sign matrices).

Hisotry: ASM conjecture. Robbins-Rumsey (1983)

$$|A_n| = \qquad\qquad \text{as} \qquad \begin{array}{c} 1 \\ 2 \\ 7 \\ 42 \\ 7436 \\ 218348 \\ 10850216 \\ 911835460 \\ 129534272700 \\ 31095744852375 \\ 12611311859677500 \\ \vdots \end{array}$$

$$\begin{cases} \text{Zeilberger (1995)} \\ \text{Kuperberg (1996)} \end{cases}$$

Byproducts.

Def (Corner sum matrices, essential sets).

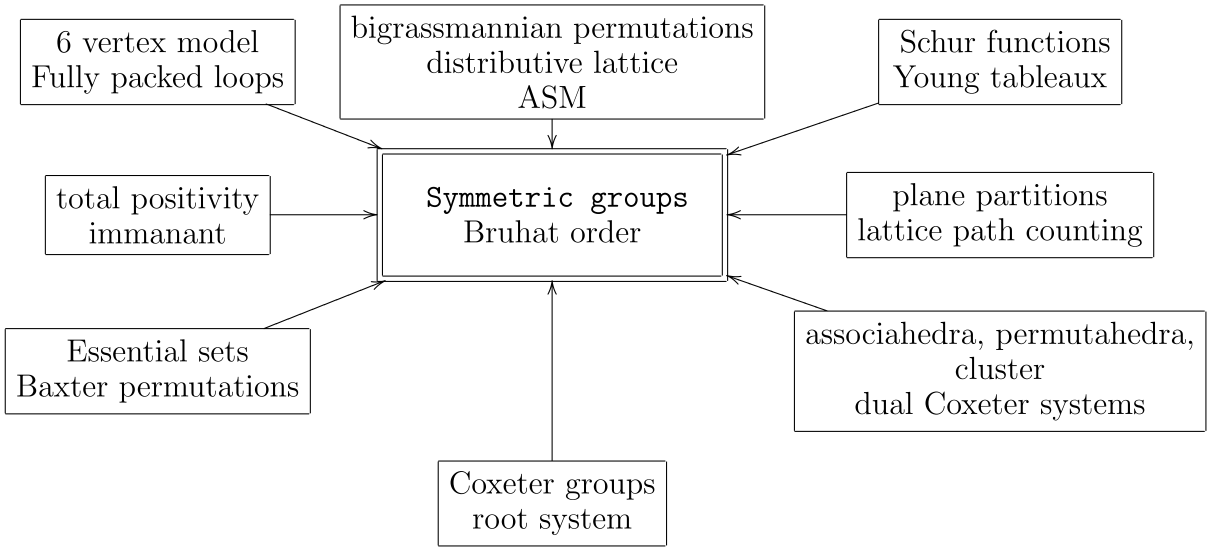
Def (bigrassmannian statistics).

Corollary (essnetial criterion for ASMs). *The following are equivalent:*

(1)

(2)

Proposition (cluster-like structure, Kobayashi).



Open problems.

- ASM polytope

- ASM and Cluster structure?

- type BC, D, affine

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