

More Combinatorics of Fulton's Essential Set

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Abstract

We develop combinatorics of Fulton's essential set particularly with a connection to Baxter permutations. For this purpose, we introduce a new idea: dual essential sets. Together with the original essential set, we reinterpret Eriksson-Linusson's characterization of Baxter permutations in terms of colored diagrams on a square board. We also discuss a combinatorial structure on local moves of these essential sets under weak order on the symmetric groups. As an application, we extend several familiar results on Bruhat order for permutations to alternating sign matrices: We establish an improved criterion of Bruhat-Ehresmann order as well as Generalized Lifting Property using bigrassmannian permutations, a certain subclass of Baxter permutations.

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1 Introduction

1.1 Essential sets

Fulton [14] introduced the *essential set* for a permutation in the course of studying Schubert polynomials and degeneracy loci. This is southwest cor-

Figure 1: essential set of 5736241

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ner of the Rothe diagram of the permutation drawn as a subset of a square board (we will give a precise definition later); For example, five white circles in Figure 1 indicate elements of the essential set of 5736241. As this naming suggests, it plays an essential role in combinatorics. The main purpose of this article is to develop such combinatorics particularly with a connection to *Baxter permutations*.

1.2 Baxter permutations

A permutation x on $\{1, 2, \dots, n\}$ is *Baxter* if whenever $1 \leq i < j < k < l \leq n$, then the following two conditions hold:

- (1) $x(i) + 1 = x(l), x(j) > x(l) \implies x(k) > x(l)$.
- (2) $x(l) + 1 = x(i), x(k) > x(i) \implies x(j) > x(i)$.

This is named after work of G. Baxter [3] in 1964. One important subsequent research is Chung-Graham-Hoggatt-Kleiman [6] on counting the number of such permutations:

$$B(n) = \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \sum_{r=0}^{n-1} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}.$$

After this work, many authors constructed new combinatorial objects which are in bijection with Baxter permutations (sometimes called *Baxter objects*) such as twin binary trees and mosaic floorplan [7, 8, 9, 11]; Separable permutations, a certain subclass of Baxter permutations, appear in the context of rectangulation and pattern avoidance (forbidden subsequences) [1, 21, 24]. Although we do not pursue any details of these topics in this article, we wish to find some explicit connection with our argument in the future.

Among these, a key result for us is a characterization of Baxter permutations in terms of essential sets by Eriksson-Linusson:

Theorem 1.1 (Eriksson-Linusson [10]). *A permutation is Baxter if and only if its essential set has at most one white corner in each row and column.*

Here “a white corner” is just the special term to indicate an element of essential sets as Fulton introduced.

1.3 Outline of the paper

Our main results consist of Theorems 3.13 (a new characterization of Baxter permutations with dual essential sets), 3.24 (Weak order and local moves), 4.6 (Essential Criterion), 4.25 (Generalized Essential Criterion) and 4.27 (Generalized Lifting Property). The first two theorems enable us to better understand Baxter permutations through our new ideas: *dual essential sets* and *essential diagram*. The other three are based on a less-known connection between essential sets and bigrassmannian permutations due to the author [17]. Interesting parts of those discussions are that we can extend some well-known Coxeter-theoretic results on permutations to *alternating sign matrices*.

In Section 2, we provide preliminaries on diagrams and symmetric groups. Section 3 gives a new characterization of Baxter permutations. In Section 4, we show how essential sets play a role also in the theory of “Bruhat order” introduced by corner sum matrices (there are many equivalent formulations of this order). We end with several remarks for our future work in Section 5.

2 Permutations and Essential sets

Throughout n is a positive integer. To avoid some triviality, we assume that $n \geq 3$. Let $[n]$ denote the set $\{1, 2, \dots, n\}$ for brevity.

2.1 Diagrams

We begin with a definition of a diagram. This is a convenient tool to develop combinatorics of essential sets in the sequel.

Definition 2.1. A *diagram* is a subset of $[n] \times [n]$. Call an element of a diagram a *rook*.

Terms on the set theory as well as matrix theory are useful: The *empty* diagram is one with no rooks. Two diagrams are *disjoint* if they do not have any rooks at the same position. The *sum* of diagrams is the one with all rooks of the diagrams. Often, we visibly express a diagram as $\{(1, 3), (3, 1)\} =$

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, for example.

Definition 2.2. A *colored diagram* is a subset of $[n] \times [n] \times C$ with a nonempty set C .

Note: formally, we allow two rooks of distinct colors to be at the same position. However, we rarely deal with such diagrams below.

Again, we often represent colored diagrams as

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	○	●

with two colors

$C = \{\circ, \bullet\} = \{\text{white}, \text{black}\}$.

Definition 2.3. Say a diagram is *noncrossing* if there is at most one rook in each row and column. A noncrossing diagram is *maximal* if there is exactly one rook in each row and column.

We will treat mainly three kinds of colored rooks: $\circ$ (*white*), $\bullet$ (*black*) and $*$ (*star*). Each rook has a corresponding role: $\circ$ for essential sets, $\bullet$ for dual essential sets and $*$ for permutation matrices. This distinction will enable us to better understand combinatorics of Baxter permutations later.

2.2 Permutations

By S_n we mean the symmetric group on $[n]$. Unless otherwise specified, letters v, w, x, y, z mean elements of S_n below. We often use one-line notation: for example, $x = 312$ means $x(1) = 3$, $x(2) = 1$ and $x(3) = 2$. The *permutation matrix* for $x \in S_n$ is the n by n matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if $j = x(i)$ and $a_{ij} = 0$ otherwise. We express such matrices by placing star rooks (indicating positions of 1s) on the square board. Whenever there is no confusion, we use the same symbol x to mean such diagram. For example,

$312 =$

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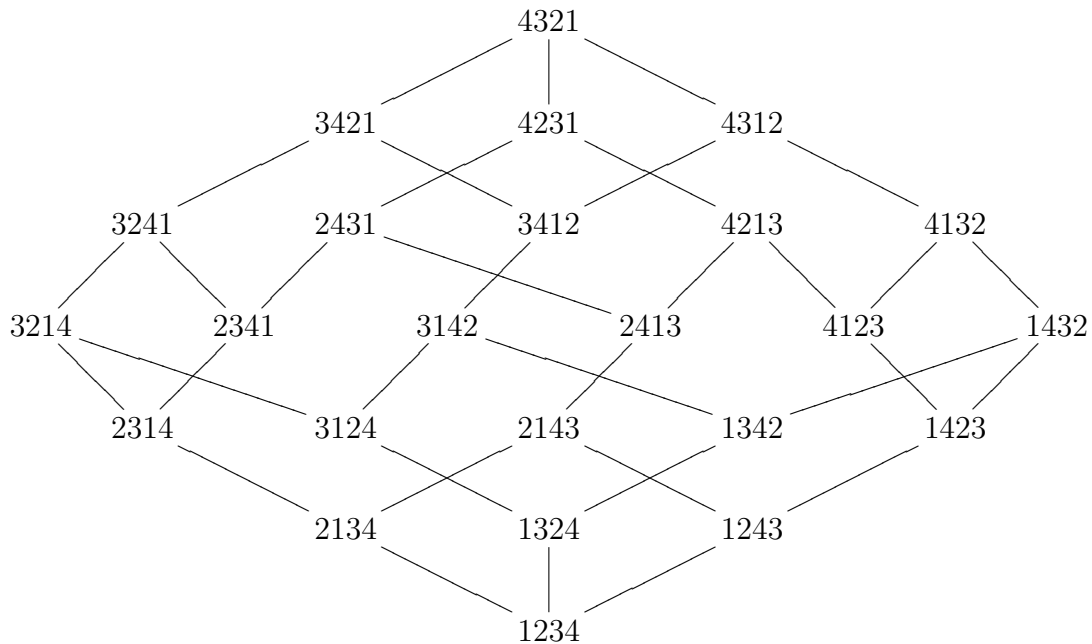
. Clearly, those diagrams are maximal by construction.

Remark 2.4. It is more common to use black dots $\bullet$ for 1s in permutation matrices. In this article, we reserve this rook for dual essential sets (Definition 3.6).

2.3 Weak orders

To investigate Baxter permutations in the next section, we have to mention *weak orders*. Before giving a precise definition, we prepare several terms on permutations: As before, consider a permutation x on $[n]$. Say a pair $(i, j) \in [n] \times [n]$ is an *inversion* of x if $i < j$ and $x(i) > x(j)$. Define $\ell(x)$ to be the number of inversions of x . Say i is a *right descent* of x if $x(i) > x(i+1)$; otherwise it is a *right rise* of x . Set $D_R(x) := \{i \in [n-1] \mid x(i) > x(i+1)\}$, the set of right descents.

Figure 2: Right weak order on S_4



Definition 2.5. Let $x, y \in S_n$ and s_i denote the adjacent transposition interchanging i and $i+1$. By $x \triangleleft_R y$, we mean $y = xs_i$ and $y(i) > y(i+1)$ for some i . Define *right weak order*: $x \leq_R y$ if there exists a chain $x \triangleleft_R x_1 \triangleleft_R x_2 \triangleleft_R \cdots \triangleleft_R y$.

Figure 2 illustrates this partial order on S_4 . This is in fact graded with the rank function ℓ . Define *left weak order* $x \leq_L y$ by $x^{-1} \leq_R y^{-1}$. A *left descent*, *left rise* and $D_L(x)$ are similarly defined (by replacing x with x^{-1}).

2.4 Essential sets

The *Rothe diagram* for x is the set

$$D(x) = \{(i, j) \in [n-1]^2 \mid i < x^{-1}(j) \text{ and } j < x(i)\}.$$

Definition 2.6. (Fulton [14]) The *essential set* of $x \in S_n$ is

$$\text{Ess}(x) = \{(i, j) \in [n-1]^2 \mid i < x^{-1}(j), j < x(i), x(i+1) \leq j, x^{-1}(j+1) \leq i\}.$$

The *essential diagram* for x is the natural rook placement representation for $\text{Ess}(x)$ with a white rook $\circ$ (keeping Fulton's term *white corner* in mind).

Example 2.7. Given x , how can we find $D(x)$ and $\text{Ess}(x)$? There is an easy

algorithm. Here is an example: let $x = 3142 =$

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. Now, kill

all positions below *s and to the right:

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*			
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	*		

. The set of survived

positions is the Rothe diagram for x . Its southeast corner is the corresponding

essential set: $\text{Ess}(3142) =$

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	○	

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3 Baxter permutations

As mentioned in Baxter permutations have rich combinatorial structures. Here we study these by introducing some new ideas: dual essential sets (and minimal-noninversion-rise sets).

3.1 Definition

Definition 3.1. A permutation x is *Baxter* if whenever $1 \leq i < j < k < l \leq n$, then the following two conditions hold:

$$(1) \quad x(i) + 1 = x(l), x(j) > x(l) \implies x(k) > x(l).$$

$$(2) \quad x(l) + 1 = x(i), x(k) > x(i) \implies x(j) > x(i).$$

Remark 3.2. We can equivalently define these permutations in terms of *generalized pattern avoidance* (according to Babson-Steingrimsson [2]) as follows: Say x *avoids 3-14-2* if there do not exist any integers (i, j, k, l) such that $i < j < k < l$, $x(k) < x(i) < x(l) < x(j)$ and $k = j + 1$. Similarly, x *avoids 2-41-3* if there do not exist any integers (i, j, k, l) such that $i < j < k < l$, $x(j) < x(l) < x(i) < x(k)$ and $k = j + 1$. Then it is the fact that x is Baxter if and only if x avoids both 3-14-2 and 2-41-3.

Let w_0 denote the reverse permutation on S_n : $i \mapsto n - i + 1$. The following group-theoretic operations induce the symmetry of matrices.

- $x \mapsto x^{-1}$: transpose.
- $x \mapsto xw_0$: reading rows backwards.

- $x \mapsto w_0x$: reading columns backwards.

Fact 3.3 (Guibert-Linusson [16, p.158]). The following are equivalent:

- (1) x is Baxter.
- (2) x^{-1} is Baxter.
- (3) xw_0 is Baxter.
- (4) w_0x is Baxter.

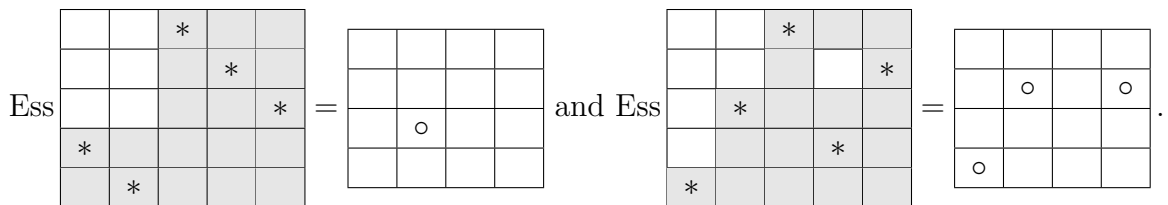
This equivalence will be helpful for the discussion below.

Example 3.4. 34512 is Baxter while 35241 is not because of the 2-41-3 pattern 3 5 2 4 1.

As mentioned in Introduction, Eriksson-Linusson found a powerful characterization of Baxter permutations in term of essential sets:

Theorem 3.5 (Eriksson-Linusson [10]). *A permutation is Baxter if and only if its essential set has at most one white corner in each row and column.*

For example,



This gives another proof of that 34512 is Baxter while 35241 is not because there are two rooks in the second row. As we see later, 34512 is an example of *bigrassmannian* permutations; every permutation in this class has the only one white rook.

3.2 Dual essential sets

Next we introduce a new object, the *dual essential set*; to best of our knowledge, it has never appeared before in the literature although the idea is simple. The *dual Rothe diagram* of x is

$$D'(x) = \{(i, j) \in [n-1]^2 \mid i \geq x^{-1}(j) \text{ and } j \geq x(i)\}.$$

Definition 3.6. The *dual essential set* of x is

$$\text{Ess}'(x) = \{(i, j) \in [n-1]^2 \mid i \geq x^{-1}(j), j \geq x(i), x(i+1) > j, x^{-1}(j+1) > i\}.$$

Remark 3.7. Some authors call any of xw_0 or w_0x the (order-)dual of x . Do not confuse with it and our term, *dual* essential sets; $\text{Ess}'(x)$ is *not* equal to any of $\text{Ess}(xw_0)$ nor $\text{Ess}(w_0x)$ (Propositions 4.1 and 4.2 will clarify the duality in our sense). Together with $\text{Ess}(x)$, it plays a supporting role to reinterpret Eriksson-Linusson's result.

We represent $\text{Ess}'(x)$ as a colored diagram with a black rook $\bullet$.

For example, let $x = 2413$. Kill all positions above stars and to the left

exclusively as

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. Moreover, exclude the n -th row and column;

this point is a little different from $D(x)$. Survived boxes are $D'(x)$. Again, its

southeast corner is $\text{Ess}'(x)$: $\text{Ess}'(2413) =$

	$\bullet$	
	$\bullet$	

. Unlike a white rook, a black rook and a star can be at the same position.

Proposition 3.8. For all $x \in S_n$ and $(i, j) \in [n-1]^2$, we have:

- (1) $(i, j) \in \text{Ess}(x) \iff (j, i) \in \text{Ess}(x^{-1})$.
- (2) $(i, j) \in \text{Ess}'(x) \iff (j, i) \in \text{Ess}'(x^{-1})$.
- (3) $(i, j) \in \text{Ess}'(x) \iff (n-i, j) \in \text{Ess}(xw_0)$.
- (4) $(i, j) \in \text{Ess}'(x) \iff (i, n-j) \in \text{Ess}(w_0x)$.

Proof. We give only a sketch of (3): check all of the following four equivalence:

$$\begin{aligned} i \geq x^{-1}(j) &\iff n-i < (xw_0)^{-1}(j). \\ j \geq x(i) &\iff (xw_0)(n-i+1) \leq j. \\ x(i+1) > j &\iff j < (xw_0)(n-i). \\ x^{-1}(j+1) > i &\iff (xw_0)^{-1}(j+1) \leq n-i. \end{aligned}$$

□

Proposition 3.9. For each $x \in S_n$, $\text{Ess}(x)$ and $\text{Ess}'(x)$ are disjoint.

Proof. Rooks of $\text{Ess}(x)$ and $\text{Ess}'(x)$ would tell descent (rise) sets of x in the following sense: If $(i, j) \in \text{Ess}(x)$, then $i \in D_R(x)$ and $j \in D_L(x)$ as $x(i) > j \geq x(i+1)$ and $x^{-1}(j) > i \geq x^{-1}(j+1)$. Analogously, if $(i, j) \in \text{Ess}'(x)$, then $i \notin D_R(x)$ and $j \notin D_L(x)$ as $x(i) \leq j < x(i+1)$ and $x^{-1}(j) \leq i < x^{-1}(j+1)$. □

Definition 3.10. Set $E(x) := \text{Ess}(x) \cup \text{Ess}'(x)$. Call $E(x)$ the *essential diagram* of x .

Proposition 3.11. For each $i \in D_R(x)$, there exists j such that $(i, j) \in \text{Ess}(x)$.

Proof. See Fulton [14, Proposition 9.18, p.413]. $\square$

Observation 3.12. Similarly, for each $j \in D_L(x)$, there exists i such that $(i, j) \in \text{Ess}(x)$. Dually, for each $i \notin D_R(x)$, there exists j such that $(i, j) \in \text{Ess}'(x)$ and for each $j \notin D_L(x)$, there exists i such that $(i, j) \in \text{Ess}'(x)$. Consequently, $E(x)$ contains at least one rook in each row and column.

Now we come to a more precise interpretation of Eriksson-Linusson. This is a key idea in our discussion below.

Theorem 3.13. *The following are equivalent:*

- (1) x is Baxter.
- (2) $\text{Ess}(x)$ is noncrossing.
- (3) $\text{Ess}'(x)$ is noncrossing.
- (4) $E(x)$ is noncrossing.
- (5) $E(x)$ is maximal.

Theorem 3.5 already shows (1) $\iff$ (2). We show (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (2).

Proof. (2) $\implies$ (3): Suppose $\text{Ess}(x)$ is noncrossing and hence x is Baxter. It follows from Fact 3.3 that xw_0 and w_0x are both Baxter so that $\text{Ess}(xw_0)$ and $\text{Ess}(w_0x)$ are noncrossing. As a result, Proposition 3.8 (3) and (4) assert that $\text{Ess}(x)$ is noncrossing. (3) $\implies$ (4): Suppose $\text{Ess}'(x)$ is noncrossing. Using Proposition 3.8 again, $\text{Ess}(xw_0)$ must be noncrossing so that xw_0 is Baxter. Hence so is x , i.e., $\text{Ess}(x)$ is noncrossing. The proof of Proposition 3.9 shows not only $\text{Ess}(x)$ and $\text{Ess}'(x)$ are disjoint but also they do not contain any rook in a common row nor column. Thus $E(x) = \text{Ess}(x) \cup \text{Ess}'(x)$ is noncrossing. (4) $\implies$ (5): Note that the number of rooks on $E(x)$ is at least $n - 1$. If $E(x)$ is noncrossing, then it must be exactly $n - 1$. Hence $E(x)$ is maximal. (5) $\implies$ (2): If $E(x)$ is maximal, then it is trivially noncrossing. Therefore so is $\text{Ess}(x)$ ($\subseteq E(x)$). $\square$

3.3 Maximal-inversion-descent sets

For every Baxter permutation x of size n , we constructed the 2-colored rook placement $E(x)$ on $[n - 1] \times [n - 1]$. This is necessarily maximal, i.e., there is exactly one rook in each row and each column. Moreover, $|D_R(x)|$ ($= |D_L(x)|$) tells the number of white rooks of $E(x)$. However, it takes some efforts to draw $E(x)$, in particular for larger n . Here we present an easier way to do this only from one-line notation of Baxter permutations.

Definition 3.14 (Min-Park [19, Definition 2.3]). The *maximal-inversion-descent set* of x is

$$M(x) = \{(i, b_i) \in [n-1]^2 \mid i \in D_R(x), b_i = \max\{x(k) \mid i < k \text{ and } x(i) > x(k)\}\}.$$

(This is $MID(x)$ in their notation).

Let us compute $M(5736241)$: $b_2 = \max\{3, 6, 2, 4, 1\} = 6$, $b_4 = \max\{2, 4, 1\} = 4$ and $b_6 = \max\{1\} = 1$. Thus, $M(x) = \{(2, 6), (4, 4), (6, 1)\}$. These are indeed white rooks:

Fact 3.15 ([19, Proposition 3.4, Theorem 3.5]). For all x , we have $M(x) \subseteq \text{Ess}(x)$. Moreover, if x is Baxter, then $M(x) = \text{Ess}(x)$.

Next, we introduce a certain dual object of $M(x)$; this is our original idea.

Definition 3.16. The *minimal-noninversion-rise set* of x is

$$M'(x) = \{(i, c_i) \in [n-1]^2 \mid i \notin D_R(x), c_i = \min\{x(k) \mid i < k \text{ and } x(i) < x(k)\} - 1\}.$$

Let us compute $M'(5736241)$: $c_1 = \min\{7, 6\} - 1 = 5$, $c_3 = \min\{6, 4\} - 1 = 3$ and $c_5 = \min\{4\} - 1 = 3$. Thus, $M'(5736241) = \{(1, 5), (3, 3), (5, 3)\}$. These are indeed black rooks (cf. Fact 3.15):

Proposition 3.17. For all x , we have $M'(x) \subseteq \text{Ess}'(x)$. Moreover, if x is Baxter, then $M'(x) = \text{Ess}'(x)$.

Proof. Let $(i, c_i) \in M'(x)$. We need to show that $c_i \geq x(i)$, $x^{-1}(c_i + 1) > i$, $x(i + 1) > c_i$ and $i \geq x^{-1}(c_i)$. Choose a unique k such that $k > i$ and $c_i + 1 = x(k) > x(i)$. It easily follows that $c_i \geq x(i)$ and $x^{-1}(c_i + 1) = k > i$. Next, by the minimality of c_i and $x(i) < x(i + 1)$, we have $c_i \leq x(i + 1) - 1$, i.e., $c_i < x(i + 1)$. Finally, suppose $i < x^{-1}(c_i)$. Then $x(k') = c_i$ for some $k' > i$. Now ask if $x(k') > x(i)$ or $x(k') \leq x(i)$. If $x(k') > x(i)$, then $x(k') - 1 \geq c_i$ by definition of c_i while $c_i - 1 = x(k') - 1 \geq c_i$, a contradiction. Hence we must have $x(k') \leq x(i)$. Now $c_i \geq x(i) \geq x(k') = c_i$ forces $i = k'$. But this is impossible.

For the second statement, suppose x is Baxter. We will show that $\text{Ess}'(x) \subseteq M'(x)$. Let $(i, j) \in \text{Ess}'(x)$. Then $(n - i, j) \in \text{Ess}(xw_0) = M(xw_0)$ because xw_0 is also Baxter. Hence it remains to show that $(n - i, j) \in M(xw_0) \implies (i, j) \in M'(x)$. Let $(n - i, j) \in M(xw_0)$. For simplicity, write $\beta := (xw_0)(n - i)$ and $\alpha := (xw_0)(n - i + 1)$. Note that $\beta > \alpha$ since $n - i$ is a descent of xw_0 . Note also that

$$j = \max\{(xw_0)(k) \mid n - i < k \text{ and } \beta > xw_0(k)\}.$$

To show $(i, j) \in M'(x)$, it suffices to verify that

$$x(i) < x(i+1) \text{ and } j = \min\{x(l) \mid i < l \text{ and } x(l) > \alpha\} - 1.$$

The first statement is equivalent to $\beta > \alpha$. Set

$$\begin{aligned} A &:= \{(xw_0)(m) \mid m \leq n - i \text{ and } (xw_0)(m) > \alpha\} \\ &= \{x(l) \mid i < l \text{ and } x(l) > \alpha\}. \end{aligned}$$

Make sure that A is nonempty since $\beta \in A$. Note that $j+1 \in A$ since the value $j+1$ appears before $n-i$ by definition of j , i.e., $(xw_0)^{-1}(j+1) \leq n-i$ and $j+1 > j \geq \alpha$. If $j = \alpha$, then certainly $j+1 = \min A$. Thus we assume $j > \alpha$. Assume moreover that there is $p \in A$ such that $p < j+1$ (in fact $p < j$ since $(xw_0)^{-1}(p) < n-i < (xw_0)^{-1}(j)$). Then (p, β, α, j) gives a forbidden pattern 2-41-3 for xw_0 . Since xw_0 is baxter, this is a contradiction. Thus we proved $j+1 = \min A$. $\square$

3.4 Cluster-like structure and local moves

In this subsection, we study a relation on $E(x)$ and right weak order among Baxter permutations; we will explain a motivation of “cluster-like structure” in the title in Section 5. As the discussion below involves diagram chasing in many cases, we proceed little by little with a series of lemmas. For this purpose, we introduce several notation here. Recall that we used numbers b_i and c_i to compute $E(x)$. Express dependence on x as $b_i(x)$ and $c_i(x)$ if necessary. Let $a_i(x) = \begin{cases} b_i(x) & \text{if } i \in D_R(x), \\ c_i(x) & \text{if } i \notin D_R(x). \end{cases}$ For integers $i < j$, let $[i, j]$ ((i, j)) denote a closed (an open) interval $\{k \mid i \leq k \leq j\}$ ($\{k \mid i < k < j\}$) in positive integers. We assume the following condition in lemmas below:

$$(\star) \ y = xs_i, \ y(i) > y(i+1) \text{ and } x, y \text{ are Baxter.}$$

Lemma 3.18. *Suppose $(\star)$. Then for all $k \notin \{i-1, i, i+1\}$, we have*

- (1) $k \in D_R(x) \iff k \in D_R(y)$ and
- (2) $a_k(x) = a_k(y)$.

In other words, $E(x)$ and $E(y)$ have rooks in the same color in the k -th row ($k \notin \{i-1, i, i+1\}$) at the same position.

Proof. (1) is an easy consequence of $x(j) = y(j)$ for all $j \neq i, i+1$. (2): Note that $a_k(x)$ and $a_k(y)$ are determined by sets $\{x(j) \mid j \in [k, n]\}$ and $\{y(j) \mid j \in [k, n]\}$. For $k \notin \{i-1, i, i+1\}$, these sets are equal. $\square$

Lemma 3.19. *Suppose $(\star)$. Then at least one of the following two cases occurs:*

$$(N) \ y^{-1}(y(i+1), y(i)) \subseteq [1, i-1].$$

$$(S) \ y^{-1}(y(i+1), y(i)) \subseteq [i+2, n].$$

Moreover, both cases occur if and only if $y(i) - y(i+1) = 1$.

That is, all stars between $y(i+1)$ - and $y(i)$ -th columns must lie either above the i -th row or below the $(i+1)$ -st row. The letters N and S mean North and South, respectively.

Proof. If both of (N) and (S) are false, then x contains 3-14-2 or y contains 2-41-3. This is a contradiction. Both cases occur if and only if both of $y^{-1}(y(i+1), y(i))$ and $y^{-1}(y(i+1), y(i))$ are empty if and only if $y(i) - y(i+1) = 1$. $\square$

Definition 3.20. Suppose $(\star)$.

- (1) We say that the covering relation $x \triangleleft_R y$ is *type (N_i^π)* ($i \in [n-1], \pi \in S_4$) if (N) occurs and the relative order of $y(i-1), y(i), y(i+1), y(i+2)$ is same to $\pi(1), \pi(2), \pi(3), \pi(4)$.
- (2) We say that the covering relation $x \triangleleft_R y$ is *type (S_i^π)* ($i \in [n-1], \pi \in S_4$) if (S) occurs and the relative order of $y(i-1), y(i), y(i+1), y(i+2)$ is same to $\pi(1), \pi(2), \pi(3), \pi(4)$.

Remark 3.21. When $i = 1$ or $i = n$, regard π as a permutation in S_3 ignoring “ $y(0)$ ” or “ $y(n+1)$ ” parts accordingly.

For example, $43521 \rightarrow 45321$ is type (N_2^{3421}) while $23514 \rightarrow 25314$ is of type (S_2^{2431}) .

Lemma 3.22. *Suppose $(\star)$. According to type of the covering relation $x \triangleleft_R y$ as below, we have:*

$$(N_i^{4321}) \ b_{i-1}(y) = b_{i-1}(x), \ b_i(y) = y(i+1), \ c_i(x) = y(i) - 1, \ b_{i+1}(y) = b_{i+1}(x).$$

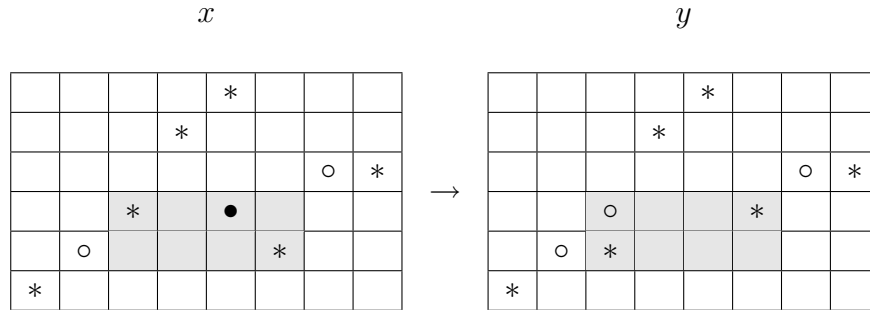
$$(S_i^{4321}) \ b_{i-1}(y) = b_{i-1}(x), \ b_i(y) = y(i) - 1, \ c_i(x) = y(i+1), \ b_{i+1}(y) = c_{i+1}(x).$$

$$(S_i^{4312}) \ b_{i-1}(y) = b_{i-1}(x), \ b_i(y) = y(i) - 1 = b_{i+1}(x), \ c_i(x) = y(i+1) = c_{i+1}(y).$$

$$(N_i^{4213}) \ b_{i-1}(y) = b_{i-1}(x) = y(i-1) - 1, \ b_i(y) = y(i+1), \ c_i(x) = y(i) - 1, \\ c_{i+1}(y) = c_{i+1}(x) = y(i).$$

In any case, $E(x)$ has a black rook and $E(y)$ has a white rook in the i -th row, respectively. Further, at most one more row is different; it must be the $(i+1)$ -st row if this is the case.

Proof. All of the proofs are diagram chasing. We prove only the first case. Consider mixed diagrams $x \cup E(x)$ and $y \cup E(y)$. As the following diagram illustrates, $b_{i-1}(y) = b_{i-1}(x)$ because $x([i, n]) = y([i, n])$; $b_i(y) = \max\{y(k) \mid i < k \text{ and } y(i) > y(k)\} = y(i+1)$ (use $y^{-1}(y(i+1), y(i)) \subseteq [1, i-1]$); $c_i(x) = \min\{x(k) \mid x(i) < x(k)\} - 1 = x(i+1) - 1 = y(i) - 1$; $b_{i+1}(y) = b_{i+1}(x)$ since $y([i+1, n]) = x([i+1, n])$.



□

Note that N_i^{4312} nor S_i^{4213} cannot occur because x, y are Baxter. To investigate other types, we make use of the following symmetry:

Lemma 3.23. *For all $z \in S_n$, we have*

$$\begin{aligned} (i, j, \circ) \in E(z) &\iff (n-i, j, \bullet) \in E(zw_0) &\iff (n-i, n-j, \circ) \in E(w_0zw_0). \\ (i, j, \bullet) \in E(z) &\iff (n-i, j, \circ) \in E(zw_0) &\iff (n-i, n-j, \bullet) \in E(w_0zw_0). \end{aligned}$$

Proof. See Proposition 3.8. □

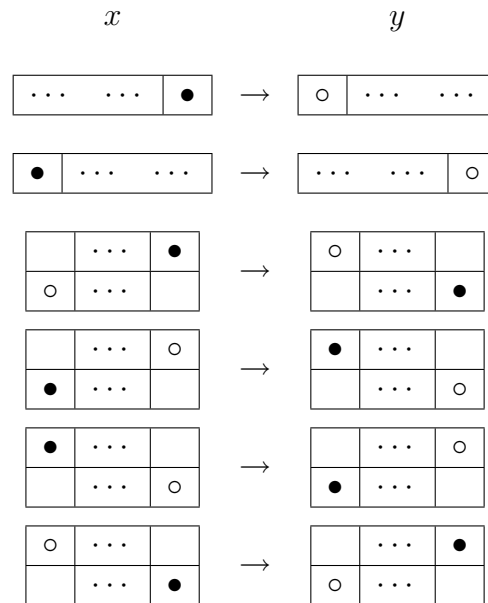
Among 24 permutations in S_4 , exactly half of them have a descent $i = 2$. Since x, y are assumed to be Baxter, it is necessary that $\pi \neq 3412, 2431, 2413$. All other cases are $\pi = 4321, 4312, 4213, 3214, 1432, 1324, 2314, 3421$ and 1423 . We already treated $\pi = 4321, 4312$ and 4213 . Observe now that

$$\begin{aligned} \{1234, 1324\} &= \{4231, 4321\}w_0, & \{2134, 2314\} &= \{4132, 4312\}w_0, \\ \{3124, 3214\} &= \{4123, 4213\}w_0, & \{1342, 1432\} &= w_0\{3142, 3214\}w_0, \\ \{3241, 3421\} &= w_0\{4132, 4312\}w_0, & \{1243, 1423\} &= w_0\{2134, 2314\}w_0. \end{aligned}$$

It turns out that these cases are reduced to one of the cases in Lemma 3.22 up to reading rows or columns backwards and changing colors if necessary (due to Lemma 3.23). We thus essentially exhausted all types.

Theorem 3.24. *Suppose $(\star)$. Then $E(x)$ has a black rook and $E(y)$ has a white rook in the i -th row, respectively. Further, at most one more row is*

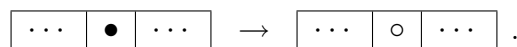
different; it must be the $(i - 1)$ - or $(i + 1)$ -st row as shown below:



If this is the case, let us say that $E(y)$ is obtained from $E(x)$ by a *local move*.

See Figure 3 for example (cf. Figure 2). A dotted line indicates a weak covering relation such that one of the two permutation is not Baxter so that the description above is not valid.

Observation 3.25. Suppose $(\star)$. Furthermore, if $y(i) - y(i + 1) = 1$ (left weak order), then $E(x)$ and $E(y)$ differ only by the color of rooks at the same position in the i -th row:



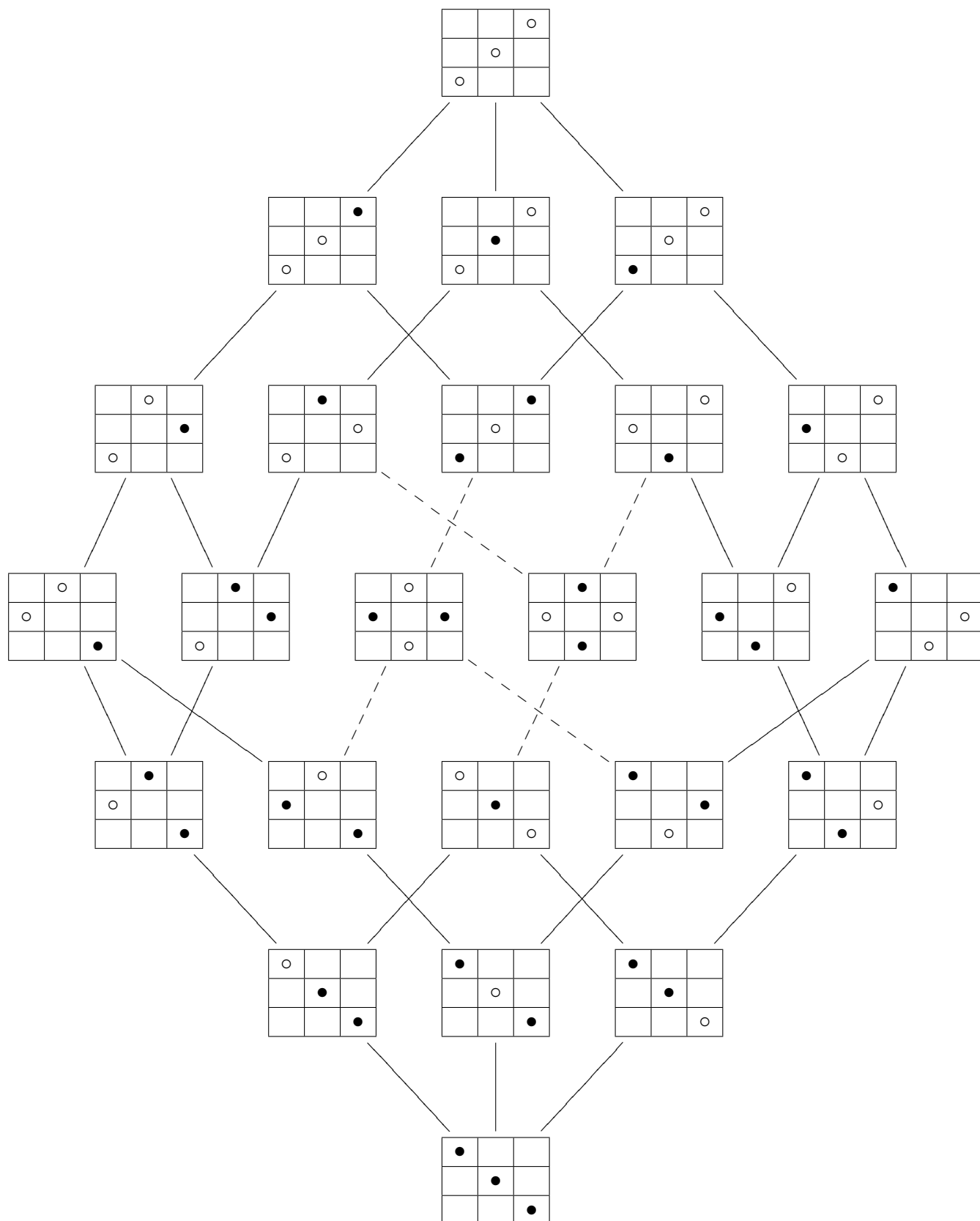
4 Bruhat order

Next, we study another partial order on S_n , *Bruhat order*, as it often appears in the theory of Coxeter groups. Essential sets play a significant role even in this setting.

4.1 Corner sum matrices

A diagram D is *ranked* if it is colored by nonnegative integers. The *restriction* of D onto a subset $A \subseteq [n]^2$ is the subdiagram $D|_A$ with ranks at only restricted positions.

Figure 3: Essential diagrams on S_4



For $x \in S_n$ (not necessarily Baxter), let $\tilde{x}(i, j) = |\{k \in [i] \mid x(k) \in [j]\}|$. This number counts star rooks in the permutation matrix x in the northwest part of (i, j) inclusively. Denote by $\tilde{x}$ the ranked diagram $(\tilde{x}(i, j))_{i,j=1}^{n-1}$ (since $\tilde{x}(i, n) = i$, $\tilde{x}(n, j) = j$ for all i, j , we simply omit the n -th row and column). Sometimes we call it the *corner sum matrix* of x for the following reason: let x_{ij} denote entries of x , that is, $x_{ij} = 1$ if $j = x(i)$ and 0 otherwise. Then $\tilde{x}(i, j) = \sum_{p \leq i, q \leq j} x_{pq}$ is literally a corner sum. It is often convenient to set $\tilde{x}(i, j) = 0$ whenever i or j is 0.

For example, if $x =$

	*		
		*	
*			
			*

,
 then $\tilde{x} =$

0	1	1
0	1	2
1	2	3

.

We now understand $\text{Ess}(x)$ in terms of $\tilde{x}$.

Proposition 4.1 (Essential conditions). *Let $x \in S_n$ and $(i, j) \in [n-1]^2$. Then we have*

- (1) $j < x(i) \iff \tilde{x}(i-1, j) = \tilde{x}(i, j).$
- (2) $i < x^{-1}(j) \iff \tilde{x}(i, j-1) = \tilde{x}(i, j).$
- (3) $x(i+1) \leq j \iff \tilde{x}(i+1, j) = \tilde{x}(i, j) + 1.$
- (4) $x^{-1}(j+1) \leq i \iff \tilde{x}(i, j+1) = \tilde{x}(i, j) + 1.$

Proof. (1): Suppose $\tilde{x}(i-1, j) = \tilde{x}(i, j)$. This is equivalent to $\sum_{p \leq i-1, q \leq j} x_{pq} = \sum_{p \leq i, q \leq j} x_{pq}$, i.e., $\sum_{q \leq j} x_{iq} = 0$. This implies that $x_{iq} = 0$ for all $q = 1, 2, \dots, j$ since each x_{iq} is either 0 or 1. Therefore there exists a unique k such that $k > j$ and $x_{ik} = 1$, in other words, $j < k = x(i)$. This argument is reversible so that the desired equivalence follows. (3): Suppose $\tilde{x}(i+1, j) = \tilde{x}(i, j) + 1$. Then $\sum_{p \leq i+1, q \leq j} x_{pq} = \left(\sum_{p \leq i, q \leq j} x_{pq}\right) + 1$, that is, $\sum_{q \leq j} x_{i+1, q} = 1$. Therefore there exists a unique k such that $k \leq j$ and $x_{i+1, k} = 1$, in other words, $x(i+1) = k \leq j$ (and vice versa). Proofs of (2) and (4) are similar: first, interchange i and j then replace x by x^{-1} . $\square$

Proposition 4.2 (Dual Essential conditions). *Let $x \in S_n$ and $(i, j) \in [n-1]^2$. Then we have*

- (1) $j \geq x(i) \iff \tilde{x}(i-1, j) = \tilde{x}(i, j) - 1.$
- (2) $i \geq x^{-1}(j) \iff \tilde{x}(i, j-1) = \tilde{x}(i, j) - 1.$
- (3) $x(i+1) > j \iff \tilde{x}(i+1, j) = \tilde{x}(i, j).$
- (4) $x^{-1}(j+1) > i \iff \tilde{x}(i, j+1) = \tilde{x}(i, j).$

Proof. This is nothing but the negation of the proposition above. $\square$

Corner sum matrices play an important role for a definition of Bruhat order:

Definition 4.3. Define *Bruhat order* $x \leq y$ in S_n if $\tilde{x}(i, j) \geq \tilde{y}(i, j)$ for all i, j (reversed inequalities).

This is a partial order graded by ℓ . As is well-known, right weak order (and left) is a indeed a suborder of this order.

Example 4.4. Comparing *all* entries of $\widetilde{13254} =$

1	1	1	1
1	1	2	2
1	2	3	3
1	2	3	3

and $\widetilde{35241} =$

0	0	1	1
0	0	1	1
0	1	2	2
0	1	2	3

yields $13254 < 35241$ in Bruhat order.

Definition 4.5. For each x , define $F(x) = \{(i, j, \tilde{x}(i, j)) \mid (i, j) \in \text{Ess}(x)\}$. Let us call $F(x)$ the *Fulton diagram* of x .

Apparently, this is the restriction of $\tilde{x}$ onto $\text{Ess}(x)$ as $F(13254) =$

	1		
			3

and $F(35241) =$

	0		1
0			

(cf. Example 4.4).

As defined above, Bruhat order is entrywise comparison of corner sum matrices. From the definition, it seems to require to know all entries of two matrices for a criterion of this order. However, our main result asserts that, to determine a weak inequality, it is enough to compare entries on the essential set for a lower element. This result provides another evidence of Fulton's naming "essential" sets from a combinatorial point of view.

Theorem 4.6. (*Essential Criterion*) *The following are equivalent:*

- (1) $\tilde{x}(i, j) \geq \tilde{y}(i, j)$ for all (i, j) .
- (2) $\tilde{x}(i, j) \geq \tilde{y}(i, j)$ for all $(i, j) \in \text{Ess}(x)$.

We postpone the proof of this result to the next subsection; To this end, we need to mention a less-known connection between essential sets and *bigrassmannian permutations* due to the author [17].

4.2 Bigrassmannian permutations

This subsection is devoted to a concrete description of bigrassmannian permutations. Despite of its importance, not many papers discuss this class; we recommend Lascoux-Schützenberger [18] and Geck-Kim [15].

Definition 4.7. Say x is *bigrassmannian* if $|D_L(x)| = |D_R(x)| = 1$; in other words, $E(x)$ has exactly one white rook. Denote by B_n the set of all bigrassmannian permutations in S_n .

Observe that every bigrassmannian permutation is Baxter.

Example 4.8.

$$B_4 = \{1243, 1324, 1342, 1423, 2134, 2314, 2341, 3124, 3412, 4123\}.$$

Following Reading [20, Section 8], let us introduce a particular parametrization of bigrassmannian permutations.

Definition 4.9. Let $\mathbf{N}$ be the set of positive integers. Consider the following index set:

$$I_n = \{(a, b, c) \in \mathbf{N}^3 \mid 1 \leq b \leq a \leq n-1 \text{ and } b+1 \leq c \leq n-a+b\}.$$

For each $(a, b, c) \in I_n$, define $J_{abc} \in B_n$ by

$$J_{abc}(i) = \begin{cases} i & \text{if } 1 \leq i \leq b-1 \\ i+c-b & \text{if } b \leq i \leq a \\ i-a+b-1 & \text{if } a+1 \leq i \leq a-b+c \\ i & \text{if } a-b+c+1 \leq i \leq n. \end{cases}$$

Note: Contrary to Reading's work in S_{n+1} , we define this in S_n .

The correspondence $J_{abc} \leftrightarrow (a, b, c)$ is indeed a bijection between B_n and I_n . To see this, just check that $|B_n| = |I_n| = \binom{n+1}{3}$ (binomial coefficient).

Remark 4.10. This parametrization (a, b, c) comes from some discussions on *monotone triangles* (certain integer arrays with entrywise order); J_{abc} is the minimum element among monotone triangles x such that " $x_{ab} \geq c$ " [20, Section 8] as explained below. We will give an explicit description on parameters (a, b, c) in terms of our main ideas, essential sets and corner sum matrices. The most technical part is Fact 4.17.

The *position* of a bigrassmannian x is (i, j) where i, j are unique elements of $D_R(x)$ and $D_L(x)$. Let $p(x) = (i, j)$. Observe $p(J_{abc}) = (a, c-1)$ since $J_{abc}(a) = a-b+c > b = J_{abc}(a+1)$ and $J_{abc}^{-1}(c-1) = a-b+c > b = J_{abc}^{-1}(c)$.

Proposition 4.11. $F(J_{abc}) = \{(a, c-1, b-1)\}$.

Proof. Recall that $\text{Ess}(J_{abc})$ consists of the only one element. Its position must be $(a, c-1)$. The rank at that position is

$$\widetilde{J_{abc}}(a, c-1) = |\{k \in [a] \mid J_{abc}(k) \in [c-1]\}| = |[b-1]| = b-1.$$

□

Lemma 4.12. *Let $x[a] = \{x(1), \dots, x(a)\}$ and $x_{a1} < x_{a2} < \dots < x_{aa}$ be the increasing arrangement of $x[a]$ (this notation has nothing to do with entries of the permutation matrix $x = (x_{ij})$). Then $x_{ab} \geq c \iff \tilde{x}(a, c-1) \leq b-1$.*

Proof. Suppose $x_{ab} \geq c$. Then, among numbers $x_{a1} < x_{a2} < \dots < x_{aa}$, at most $b-1$ elements are $\leq c-1$; in other words,

$$\tilde{x}(a, c-1) = |\{k \in [a] \mid x(k) \leq c-1\}| \leq b-1.$$

Reverse this argument to prove the converse. $\square$

Lemma 4.13. $x_{ab} \geq c \iff x \geq J_{abc}$; in other words, J_{abc} is the minimum element (under Bruhat order) of the set $\{x \in S_n \mid x_{ab} \geq c\}$. As a result, $x \geq J_{abc} \iff \tilde{x}(a, c-1) \leq b-1$.

Proof. The first equality follows from the original definition of J_{abc} [20]. For the second, combine it and the previous lemma. $\square$

This tells us that, to know Bruhat order $x \geq w$ with w bigrassmannian (and x any), it is enough to compare a rank at the only one position.

Proposition 4.14. *Bigrassmannian permutations at the same position form a chain in Bruhat order; more precisely, if $(a, b, c), (a, b', c) \in I_n$ and $b \geq b'$, then $J_{abc} \leq J_{ab'c}$.*

Proof. As discussed above, $\widetilde{J_{abc}}(a, c-1) = b-1$ and $\widetilde{J_{ab'c}}(a, c-1) = b'-1$. If $b \geq b'$, then $\widetilde{J_{ab'c}}(a, c-1) = b'-1 \leq b-1$. Lemma 4.13 now implies that $J_{ab'c} \geq J_{abc}$. $\square$

Examples for $n = 5$:

$$F(J_{323}) = F(13425) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & 1 & & \\ \hline & & & \\ \hline \end{array} \text{ and } F(J_{313}) = F(34512) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & 0 & & \\ \hline & & & \\ \hline \end{array}.$$

According to the proposition above, we immediately get $13425 < 34512$ by comparing only the rank at the $(3, 2)$ position. We can extend this idea for general permutations; we use a less-known connection between essential sets and bigrassmannian permutations.

Definition 4.15. For $x \in S_n$, let $B_n(x) = \{v \in B_n \mid v \leq x\}$.

We may rephrase Bruhat order in terms of this set:

Fact 4.16 (Lascoux-Schützenberger [18]). $B_n(x) \subseteq B_n(y) \iff x \leq y$.

Fact 4.17 (a consequence of [17, Lemma 4.2]). Let $\text{Max } P$ denote the maximal elements of a poset P . For each $x \in S_n$ and $(a, b, c) \in I_n$, we have the following:

- (1) $\{p(w) \mid w \in \text{Max } B_n(x)\} = \text{Ess}(x)$.
- (2) $(a, c-1) \in \text{Ess}(x) \implies J_{abc} \in \text{Max } B_n(x)$ and $b = \tilde{x}(a, c)$.

Lemma 4.18. *If $w \in \text{Max } B_n(x)$, then there exists some $(a, b, c) \in I_n$ such that $(a, c-1) \in \text{Ess}(x)$ and $\tilde{x}(a, c-1) = b-1$.*

Proof. Suppose $w \in \text{Max } B_n(x)$. Thanks to Fact 4.17, $w = J_{abc}$ for some $(a, b, c) \in I_n$ with $p(w) = (a, c-1) \in \text{Ess}(x)$ and $b = \tilde{x}(a, c)$. Since $(a, c-1) \in \text{Ess}(x)$, we have $\tilde{x}(a, c-1) = \tilde{x}(a, c) - 1 = b-1$ by Proposition 4.1. $\square$

Theorem 4.19. (*Essential Criterion*) *The following are equivalent:*

- (1) $x \leq y$.
- (2) $\tilde{x}(i, j) \geq \tilde{y}(i, j)$ for all $(i, j) \in \text{Ess}(x)$.

Proof. (1) $\implies$ (2) is trivial. For (2) $\implies$ (1), it is enough to show $B_n(x) \subseteq B_n(y)$. Let $v \in B_n(x)$. Choose $w \in \text{Max } B_n(x)$, say $w = J_{abc}$ with $(a, c-1) \in \text{Ess}(x)$ and $\tilde{x}(a, c-1) = b-1$, such that $v \leq w$. By assumption, $b-1 = \tilde{x}(a, c-1) \geq \tilde{y}(a, c-1)$. Now let $z = J_{ab'c}$ where $b' = \tilde{y}(a, c-1) + 1$. Then $b \geq b'$ implies $w \leq z$ because w and z have the same position. Moreover, $J_{ab'c} \leq y$ thanks to Lemma 4.12. All together, we have $v \leq w \leq z \leq y$ which concludes $v \in B_n(y)$. $\square$

$$\text{For instance, } F(13254) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & 1 & & \\ \hline & & & \\ \hline & & & 3 \\ \hline \end{array} \text{ and } \widetilde{35241}|_{\text{Ess}(13254)} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & 0 & & \\ \hline & & & \\ \hline & & & 3 \\ \hline \end{array}$$

guarantee that $13254 < 35241$.

The following is Fulton's result. We here give a rather order-theoretic proof.

Corollary 4.20. $F(x) = F(y) \implies x = y$.

Proof. $F(x) = F(y)$ is equivalent to $\tilde{x}(i, j) = \tilde{y}(i, j)$ for all $(i, j) \in \text{Ess}(x)$ and furthermore $\text{Ess}(x)$ and $\text{Ess}(y)$ are identical. Apply Theorem 4.19 for both x the lower side and y the lower side. This yields $x \leq y$ and $y \leq x$. $\square$

4.3 Alternating sign matrices

Some readers might see that Theorem 4.19 is a variant of Björner-Brenti's improved tableaux criterion [4]. In terms of Coxeter groups, $J_{abc} \in \text{Max } B_n(x)$ corresponds to the distinguished representative of minimal length of the double coset $W_{[n-1] \setminus \{c-1\}} x W_{[n-1] \setminus \{a\}}$ where W_I is the parabolic subgroup of S_n generated by $\{s_i \mid i \in I\}$ (while Björner-Brenti's argument are essentially on

one-sided cosets). So what is new here? We present an application to *alternating sign matrices* (ASMs) even though they have no longer a Coxeter group structure. An alternating sign matrix is a square matrix of 0s, 1s, -1 s for which

- the sum of the entries in each row and in each column is 1,
- the non-zero entries of each row and of each column alternate in sign.

By $\mathcal{A}_n$ we mean the set of such matrices. These are a natural generalization of permutation matrices (with the long history in enumerative combinatorics [5]); indeed it still makes sense to speak of corner sum matrices. For $A = (a_{ij}) \in \mathcal{A}_n$, define an n by n matrix $\tilde{A}$ with entries $\tilde{A}(i, j) = \sum_{p \leq i, q \leq j} a_{pq}$.

Fact 4.21 (Robbins-Rumsey [23, p.172, Lemma 1]). Let X be a square matrix of size n . Then $X = \tilde{A}$ for unique $A \in \mathcal{A}_n$ if and only if $X(i, n) = X(n, i) = i$ for all i and $X(i, j) - X(i-1, j) \in \{0, 1\}$, $X(i, j) - X(i, j-1) \in \{0, 1\}$ for all i, j .

In this way, $A \leftrightarrow \tilde{A}$ is a bijection between $\mathcal{A}_n$ and such matrices, say $\tilde{\mathcal{A}}_n$; Below we often identify them. Define *Bruhat-Ehresmann order* $A \leq B$ if $\tilde{A}(i, j) \geq \tilde{B}(i, j)$ for all i, j . Then $(\mathcal{A}_n, \leq)$ forms a distributive lattice as the Dedekind-MacNeille completion of $(S_n, \leq)$; [20] contains details of this part. Since bigrassmannian permutations are equivalently join-irreducible, $\beta(A) := |\{C \in B_n \mid C \leq A\}|$ is indeed the rank function of $\mathcal{A}_n$.

Definition 4.22. The *essential* and *dual essential set* of $A \in \mathcal{A}_n$ are

$$\begin{aligned} \text{Ess}(A) &= \{(i, j) \in [n-1]^2 \mid \tilde{A}(i, j) \text{ satisfies essential conditions}\} \text{ and} \\ \text{Ess}'(A) &= \{(i, j) \in [n-1]^2 \mid \tilde{A}(i, j) \text{ satisfies dual essential conditions}\}. \end{aligned}$$

This gives a unified treatment of essential sets (among permutations and ASMs) in terms of corner sum matrices. These sets play a role to describe a covering relation of $\mathcal{A}_n$.

Proposition 4.23. Let $A, B \in \mathcal{A}_n$. Then the following are equivalent:

- (1) $A \triangleleft B$ (a covering relation in $\mathcal{A}_n$).
- (2) There exists a unique $(i, j) \in [n]^2$ such that $\tilde{A}(i, j) = \tilde{B}(i, j) + 1$ and $\tilde{A}(k, l) = \tilde{B}(k, l)$ for all $(k, l) \in [n]^2 \setminus \{(i, j)\}$.

Proof. This easily follows from the definition of entrywise order. $\square$

Proposition 4.24. Suppose $A \triangleleft B$ and $\tilde{A}(i, j) = \tilde{B}(i, j) + 1$ as in Proposition 4.23. Then $(i, j) \in \text{Ess}(B) \cap \text{Ess}'(A)$.

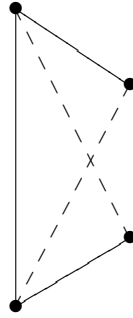


Figure 4: Lifting Property

Proof. For simplicity, let $b = \tilde{B}(i, j) = \tilde{A}(i, j) - 1$. Since both A, B are ASMs, entries of their corner sum matrices are weakly increasing with respect to rows and columns and adjacent entries differ by at most one (Fact 4.21). Hence $\tilde{B}(i-1, j) \in \{b-1, b\}$ and $\tilde{A}(i-1, j) \in \{b, b+1\}$. Since $\tilde{B}(i-1, j) = \tilde{A}(i-1, j)$, this integer must be b . Do the same argument for all other three entries adjacent to b : we obtain $\tilde{A}(i, j-1) = \tilde{B}(i, j-1) = b$ and $\tilde{A}(i+1, j) = \tilde{B}(i+1, j) = b+1 = \tilde{B}(i, j+1) = \tilde{A}(i, j+1)$. Conclude that $(i, j) \in \text{Ess}(B) \cap \text{Ess}'(A)$. $\square$

Theorem 4.25 (Generalized Essential Criterion). *Let $A, B \in \mathcal{A}_n$. Then the following are equivalent:*

- (1) $A \leq B$.
- (2) $A(i, j) \geq B(i, j)$ for all $(i, j) \in \text{Ess}(A)$.

Proof. (Sketch) Let $B_n(A) = \{C \in B_n \mid C \leq A\}$. In $\mathcal{A}_n$, we have $B_n(A) \subseteq B_n(B) \iff A \leq B$ as the Dedekind-MacNeille completion of S_n [18, 20]. Also, we can show $\text{Ess}(A) = \{p(w) \mid w \in \text{Max } B_n(A)\}$ from Proposition 4.24. So $A \leq B$ can be reduced to just showing a family of inequalities of bigrassmannian permutations as we described. $\square$

4.4 Generalized Lifting Property

We show another application of essential sets.

Fact 4.26 (Lifting Property, Figure 4). Suppose $x < y$ in S_n , $\ell(xs_i) < \ell(x)$ and $\ell(ys_i) < \ell(y)$. Then $x \leq ys_i$ and $xs_i \leq y$.

Now construct involutions $\tilde{r}_{ij} : \tilde{\mathcal{A}}_n \rightarrow \tilde{\mathcal{A}}_n$ and $r_{ij} : \mathcal{A}_n \rightarrow \mathcal{A}_n$ as an analogy of simple reflections inducing covering relations.

Let E_{ij} be the matrix with the (i, j) -entry 1 and all others 0. Define

$$\tilde{r}_{ij}\tilde{A} = \begin{cases} \tilde{A} + E_{ij} & \text{if } (i, j) \in \text{Ess}(A) \\ \tilde{A} - E_{ij} & \text{if } (i, j) \in \text{Ess}'(A) \\ \tilde{A} & \text{otherwise} \end{cases}.$$

Equivalently, this is the following trichotomy: either $\beta(r_{ij}A) < \beta(A)$ or $\beta(r_{ij}A) > \beta(A)$ or $\beta(r_{ij}A) = \beta(A)$. Define $r_{ij}A$ to be the ASM corresponding to $\tilde{r}_{ij}\tilde{A}$. Observe that $\tilde{r}_{ij}^2\tilde{A} = \tilde{A}$ and $r_{ij}^2A = A$.

Theorem 4.27 (Generalized Lifting Property). *Let $A < B$. Suppose $\beta(A) < \beta(r_{ij}A)$ and $\beta(r_{ij}B) < \beta(B)$. Then $A \leq r_{ij}B$ and $r_{ij}A \leq B$.*

Proof. Let $a = \tilde{A}(i, j)$ and $b = \tilde{B}(i, j)$ for simplicity. Since $A < B$, we have $a \geq b$. Due to (dual) essential conditions for A (B) at (i, j) , entries adjacent to a and b must be as follows:

$$\begin{array}{|c|c|c|} \hline & a-1 & \\ \hline a-1 & a & a \\ \hline & a & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline & b & \\ \hline b & b & b+1 \\ \hline & b+1 & \\ \hline \end{array}$$

Comparing these four entries, we have $a-1 \geq b$, i.e. $a > b$. By definition, $r_{ij}B$ is the ASM with the corner sum matrix identical to $\tilde{B}$ except the (i, j) -entry (which is $b+1$). Thus, $A \leq r_{ij}B$. Similarly, $r_{ij}A \leq B$. $\square$

Example 4.28. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Then $\tilde{A} = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 2 \\ \hline 1 & 2 & 2 \\ \hline \end{array}$ and $\tilde{B} = \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline 1 & 1 & 2 \\ \hline \end{array}$ so that $A < B$. Equivalently,

$F(A) = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline & & 2 \\ \hline & & 2 \\ \hline \end{array}$ and $\tilde{B}|_{\text{Ess}(A)} = \begin{array}{|c|c|c|} \hline 0 & & \\ \hline & & 1 \\ \hline & & 2 \\ \hline \end{array}$ shows that $A < B$. Observe

that $(2, 3) \in \text{Ess}(B) \cap \text{Ess}'(A)$. As a consequence, $\tilde{r}_{23}\tilde{A} = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 2 & 2 \\ \hline \end{array}$ and

$\tilde{r}_{23}\tilde{B} = \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 0 & 1 & 2 \\ \hline 1 & 1 & 2 \\ \hline \end{array}$ yields $r_{23}A \leq B$ as well as $A \leq r_{23}B$.

5 Concluding remarks

We close the article with recording several ideas for our future work.

- Here we explain why we used the term “cluster-like structure” for the title of 3.4. As in the theory of Coxeter systems, let S be the set of adjacent transpositions (simple reflections) and T the set of transpositions (reflections) in the underlying Coxeter group $W = S_n$. Let c be a Coxeter element, i.e., a product of all distinct simple reflections. Let x be a c -sortable element with c -sorting word $a = a_1 a_2 \cdots a_k$; for details of undefined terminology, see [22]. If $s \in S$ occurs in a , then the *last reflection* for s in x is $a_1 a_2 \cdots a_{j-1} a_j a_{j-1} \cdots a_2 a_1$ where a_j is the rightmost occurrence of s in a . If s does not occur in a , then the last reflection for s in x is the formal negative $-s$ (as an element of the formal negative $-S$). Let $\text{cl}_c(x)$ be the set of last reflections of x . This is a subset of $T_{\geq -1} := T \cup (-S)$ of cardinality $n - 1$. Such sets $\{\text{cl}_c(x)\}$ have some nice combinatorial structures under weak order (called *c-clusters*); Clusters are one of important topics in recent representation theory with combinatorics of root systems and Bruhat order introduced by Fomin-Zelevinsky [13]. The natural one-to-one correspondence between S and each $\text{cl}_c(x)$ (and Eriksson-Linusson’s characterization) simply leads us to the idea of essential diagrams for Baxter permutations. We should mention that we are also inspired by the following work: Reading [21] on rectangulation, pattern avoidance and Baxter permutations and Fomin-Kirillov [12] on an rc-graph (a pipe dream) in the theory of *Schubert polynomials*.
- Fulton [14] and Eriksson-Linusson [10] studied (and characterized) essential sets of permutations in various classes such as *321-avoiding*, *dominant*, *vexillary* and *grassmannian*. As an analogy, what can say about essential diagrams for these permutations?
- As discussed above, it is possible to extend the definition of essential diagrams for alternating sign matrices. For example, $E \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$

○	●
●	○

with the corner sum matrix

0	1
1	1

. A significant difference from permutations is that white and black rooks now can be in the same row or column. We should be able to investigate these diagrams since we know all covering relations of ASMs (Proposition 4.24).

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