q-DETERMINANT, q-VANDERMONDE AND SIGNED BIGRASSMANNIAN POLYNOMIALS

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ABSTRACT. It is always interesting to ask what a q-analog of something is. In studying the poset structure of alternating sign matrices, I came up with a series of three ideas as in the title. As a main theorem, I show an expression of signed bigrassmannian polynomials.

CONTENTS

- enumerative combinatorics
- Coxeter group
- distributive lattice
- linear algebra

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1. Introduction

For $w \in S_n$ $(n \geq 2)$, say (i, j) is an *inversion* of w if

$$i < j \text{ and } w(i) > w(j).$$

The *length* of w is

$$\ell(w) = |\{(i, j) \mid i < j \text{ and } w(i) > w(j)\}|.$$

The *sign* of w is $(-1)^{\ell(w)}$.

In Linear Algebra, we have seen that

$$\sum_{w \in S_n} (-1)^{\ell(w)} = 0.$$

$q=1$

What is a q-analog of this? One answer is this:

$$\sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)} = \prod_{1 \leq i < j \leq n} (1 - q^{j-i})$$

where

$$\beta(w) = \sum_{i,j:\text{inversion of } w} (j - i)$$

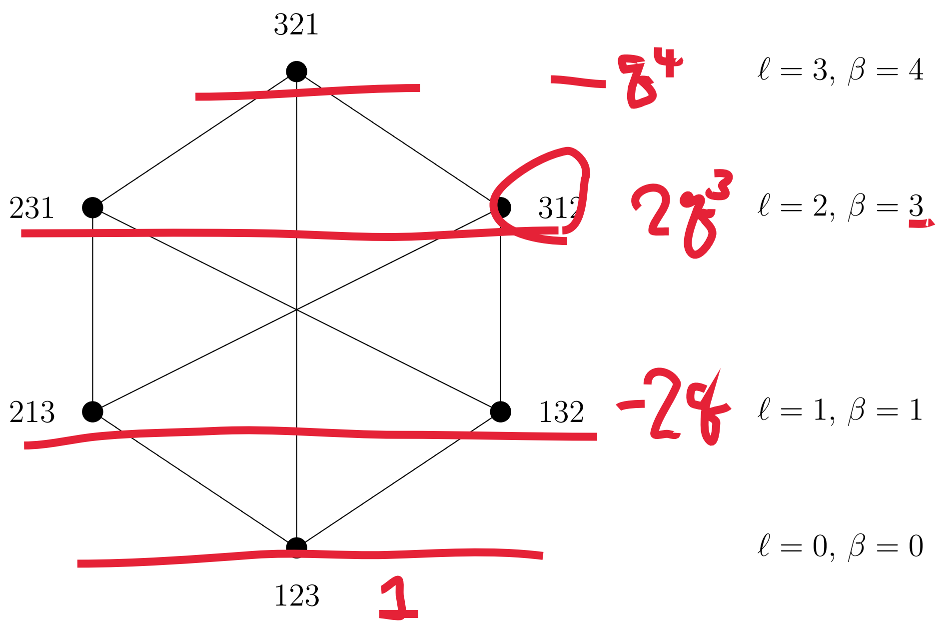
is the *bigrassmannian statistics*. For convenience, let

$$B_n(q) = \sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)}.$$

Main theorem (Kobayashi 2015, signed bigrassmannian polynomials)**.** For $n \geq 2$,

$$B_n(q) = \prod_{1 \leq i < j \leq n} (1 - q^{j-i})$$

FIGURE 1. S_3



Example.

$$w = 312,$$
$$\ell(w) = 2,$$
$$\beta(w) = (3-1) + (3-2) = 3.$$

$$B_3(q) = 1 - 2q + 2q^3 - q^4 = (1-q)^2(1-q^2).$$

How to compute $B_n(q)$? $\to$ use "det"

2. q-DETERMINANT

Fact 2.1 (Kobayashi 2011)**.**

$$\beta(w) = \frac{1}{2}\sum_{i=1}^{n}(w(i) - i)^2.$$

Definition 2.2. The *q-determinant* of $A = (a_{ij})$ is

$$\det_q(a_{ij}) = \det(q^{(j-i)^2/2}a_{ij}).$$

For example,

$$\det_q\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & q^{1/2} & q^{4/2} \\ q^{1/2} & 1 & q^{1/2} \\ q^{4/2} & q^{1/2} & 1 \end{pmatrix}.$$

We can rephrase

$$\sum_{w\in S_n}(-1)^{\ell(w)} = 0$$

as

$$\det(1)_{i,j=1}^{n} = 0.$$

A q-analog of this is

$$\begin{aligned}\det_q(1)_{i,j=1}^{n} &= \sum_{w\in S_n}(-1)^{\ell(w)}q^{(w(1)-1)^2/2}\cdots q^{(w(n)-n)^2/2} \\ &= \sum_{w\in S_n}(-1)^{\ell(w)}q^{\beta(w)} = B_n(q).\end{aligned}$$

How to compute $\det_q(1)$? $\to$ Vandermonde

3. q-Vandermonde

Classical Vandermonde:

$$\det(x_i^{j-1})_{i,j=1}^n = \prod_{1\le i<j\le n}(x_j - x_i).$$

Lemma 3.1 (q-Vandermonde)**.**

$$\det_q(x_i^{j-1})_{i,j=1}^n = \prod_{1\le i<j\le n}(x_j - q^{j-i}x_i).$$

For example,

$$\det\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = (x_2-x_1)(x_3-x_1)(x_3-x_2),$$

$$\det_q\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = (x_2-qx_1)(x_3-q^2x_1)(x_3-qx_2).$$

Fact 3.2 (Bressoud)**.**

$$\prod_{1\le i<j\le n}(1-q^{j-i}) = \sum_{w\in S_n}(-1)^{\ell(w)}q^{\sum_{i,j:\text{inversion of } w}(j-i)}$$

Let $x_i = 1$ for all i in q-Vandermonde. Then

$$\det_q(1) = \prod_{1\le i<j\le n}(1-q^{j-i}) = \sum_{w\in S_n}(-1)^{\ell(w)}q^{\sum_{i,j:\text{inversion of } w}(j-i)}.$$

B(w)

This is nothing but $B_n(q)$, as required.

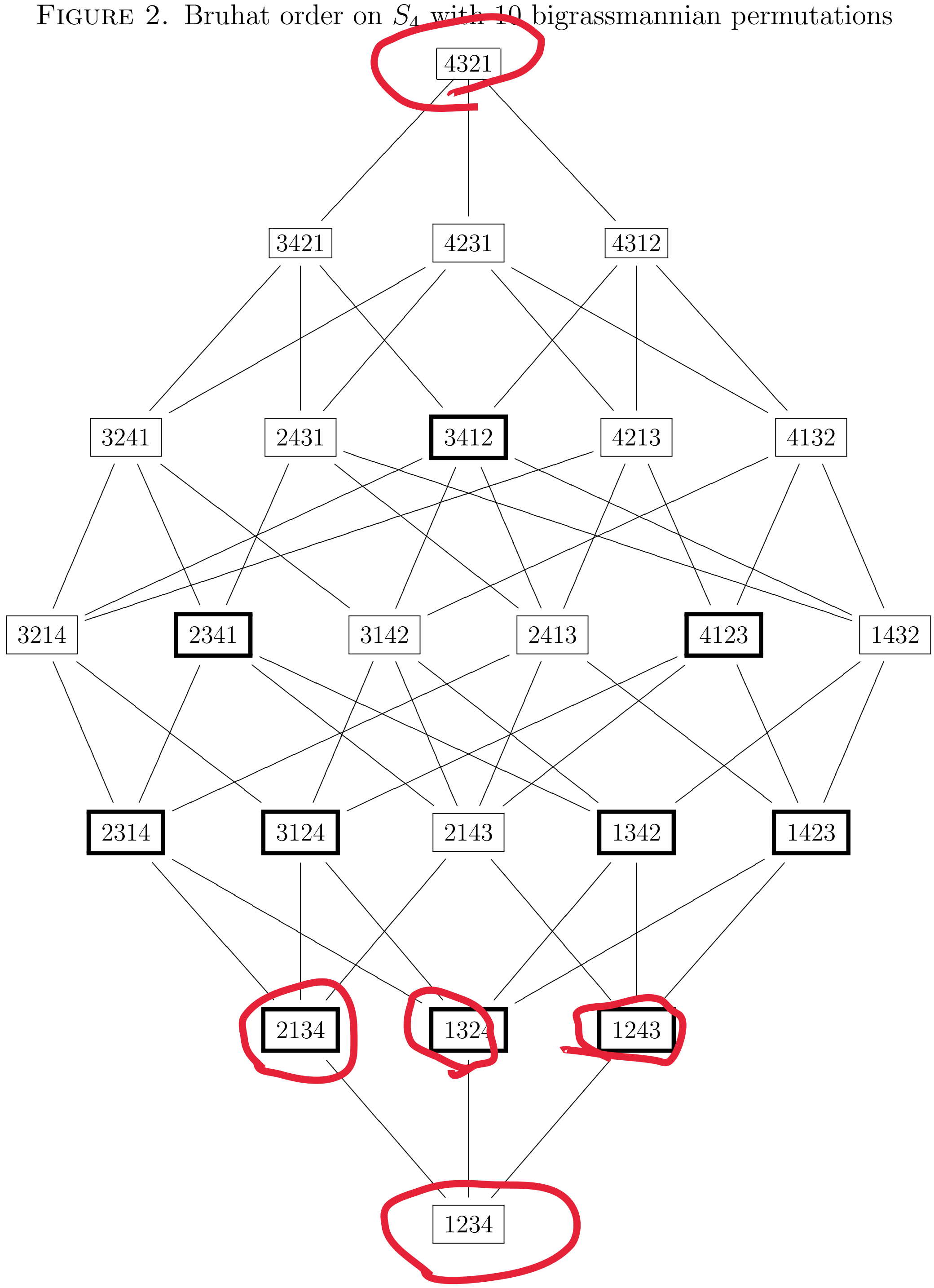
4. WHY SIGNED BIGRASSMANNIAN POLYNOMIALS?

Both $\ell(w), \beta(w)$ play a crucial role for the poset structure of S_n as follows.

Definition 4.1. Say $w \in S_n$ is *bigrassmannian* if there exists a unique pair $(i, j) \in \{1, 2, \dots, n-1\}^2$ such that $w^{-1}(i) > w^{-1}(i+1)$ and $w(j) > w(j+1)$.

Definition 4.2. Define *Bruhat order* $\leq$ on S_n as the transitive closure of the following binary relation: $v \to w$ meaning $w = vt_{ij}$, for some $i < j$, t_{ij} a transposition and $\ell(v) < \ell(w)$.

FIGURE 2. Bruhat order on S_4 with 10 bigrassmannian permutations



Let

$$B(w) = \{u \in S_n \mid u \le w \text{ and } u \text{ is bigrassmannian}\}$$

and $\beta(w) = |B(w)|$. This coincides with the $\beta(w)$ defined before.

Definition 4.3. Let P be a poset and $w \in P$. Say w is *join-irreducible* if

(1) w is not the minimum of P.
(2) $w = u_1 \vee \cdots \vee u_k \Longrightarrow w = u_i$ for some i.

Fact 4.4 (Lascoux-Schützenberger)**.** For $w \in S_n$, the following are equivalent:

(1) w is bigrassmannian.
(2) w is join-irreducible.

A finite lattice $(L, \leq, \vee, \wedge)$ is *distributive* if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

and

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in L$.

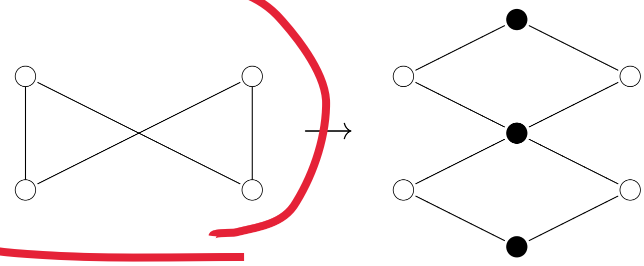
Fact 4.5. In a finite distributive lattice L, each $w \in L$ ($w \neq \min L$) can be uniquely written as

$$w = u_1 \vee \cdots \vee u_k$$

where u_i is join-irreducible.

Fact 4.6 (MacNeille)**.** If P is a finite poset, then there exists the smallest distributive lattice $L(P)$ containing P. This is the *MacNeille completion* of P.

FIGURE 3. Example of MacNeille completion



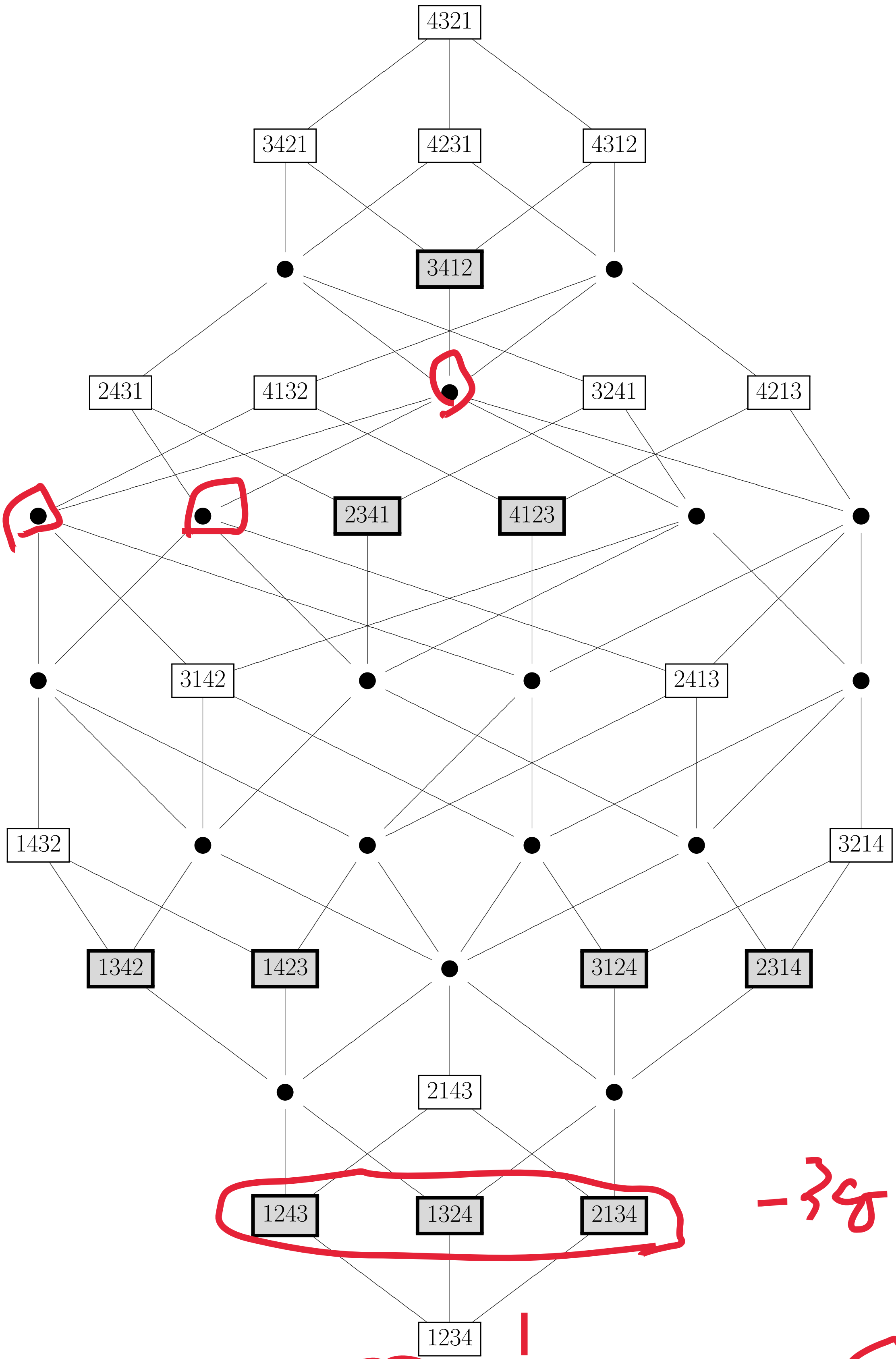
Fact 4.7. Every finite distributive lattice L is a graded poset ranked by
$$\beta(w) = |\{u \in L \mid u \leq w, u \text{ is join-irreducible}\}|.$$

Summary

$$(S_n, \leq, \ell) \xrightarrow{\text{MacNeille completion}} (L(S_n), \leq, \beta)$$

$$\sum_{w \in S_n} (-1)^{\ell(w)} = 0 \xrightarrow{q\text{-analog}} \sum_{w \in S_n} (-1)^{\ell(w)} q^{\beta(w)} = \prod_{1 \leq i < j \leq n} (1 - q^{j-i})$$

10



$-3q$

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$$B_4(q) = (1-q)^3(1-q^2)^2(1-q^3) = 1-3q+q^2+4q^3-2q^4-2q^5-2q^6+4q^7+q^8-3q^9+q^{10}$$

Thanks!

References

[1] M. Kobayashi, Enumeration of bigrassmannian permutations below a permutation in Bruhat order, Order 28 (2011), no. 1, 131-137.
[2] M. Kobayashi, Enumerative Combinatorics of determinants and signed bigrassmannian polynomials, Okayama Math. J. 57 (2015), 159-172.